# Magnitude of Metric Spaces II

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Integral Geometry and Valuation Theory, CRM Barcelona
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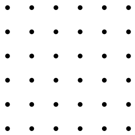

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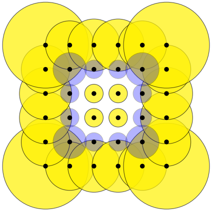

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Theorem (Meckes): Suppose $A \subset \mathbb{R}^m$.
If $\{\ddot{A}_i\}$ is a sequence of finite subsets of $A$ with $\ddot{A}_i \to A$ then $|A_i| \to |A|$.

# Homogeneous spaces and circles

Lemma (Speyer): Suppose $A$ is a homogeneous metric space.
There is a constant weighting $w$: for any fixed $a_0 \in A$

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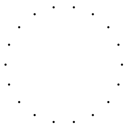
$C_\ell^n :=$ 

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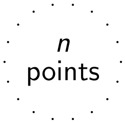
$C^n_\ell :=$ 

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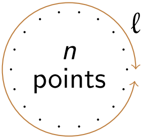
$C_\ell^n :=$ 

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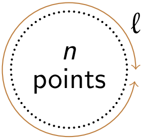

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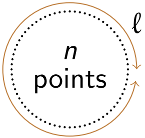


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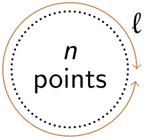


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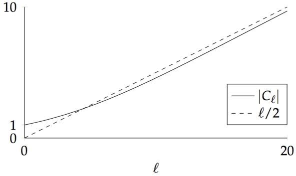

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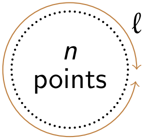


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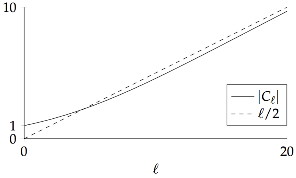

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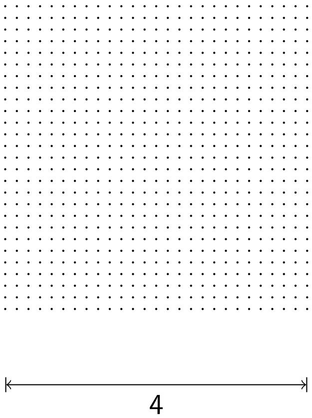

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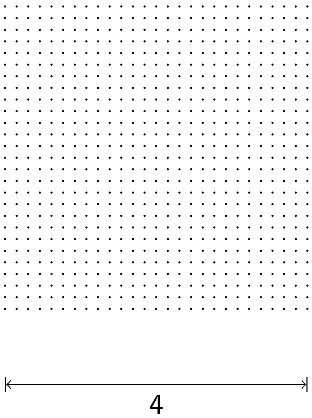

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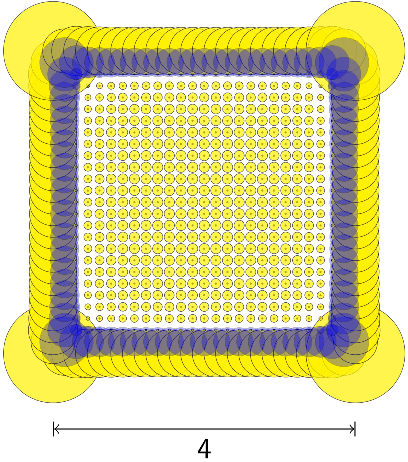

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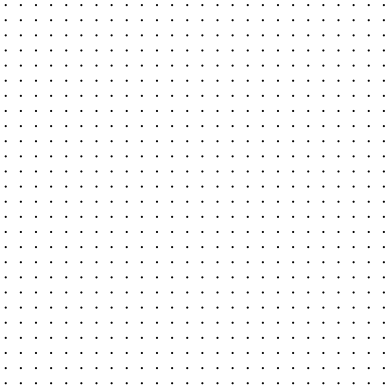

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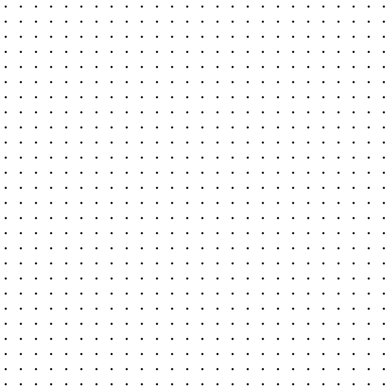

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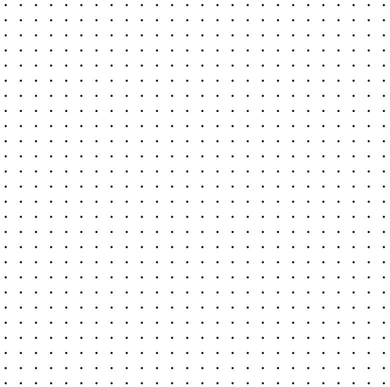

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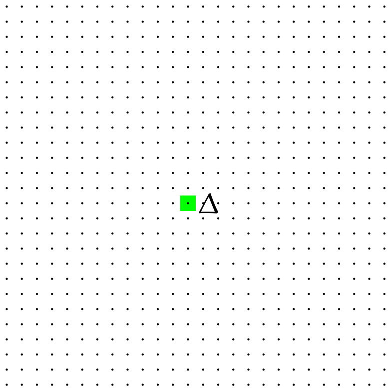

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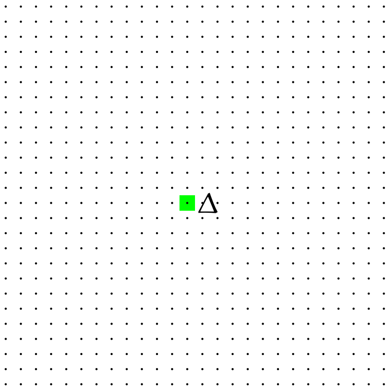

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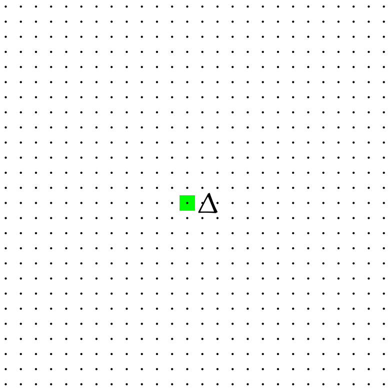

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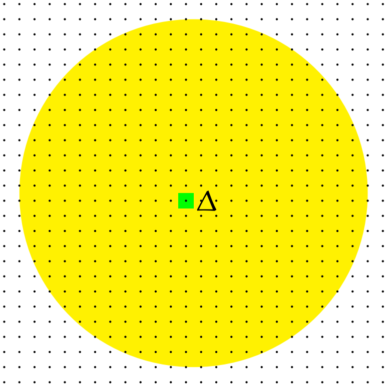


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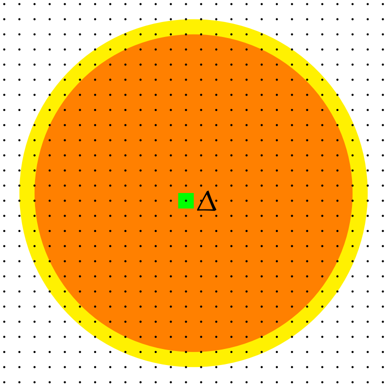


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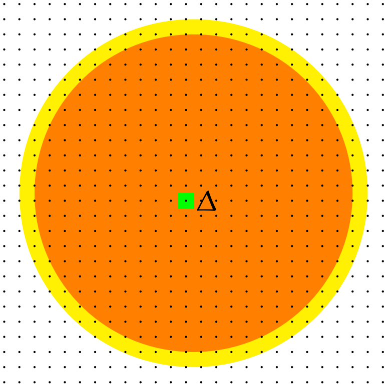


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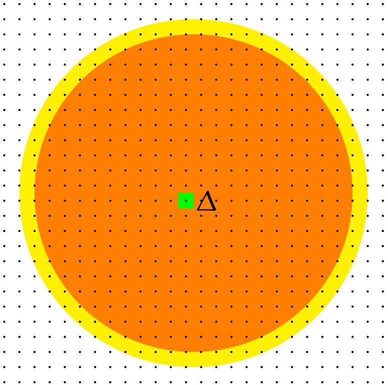


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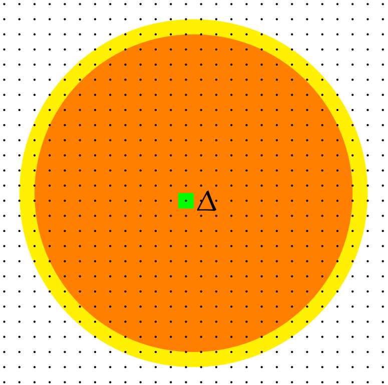


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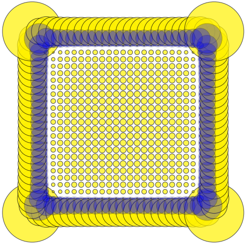
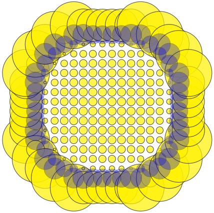
# Some calculations

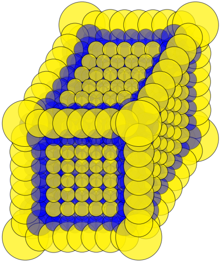
Squares:



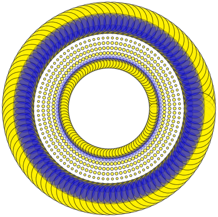
Discs:



Cubes:



Annuli:



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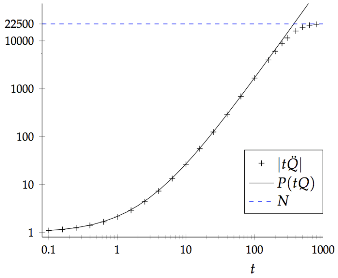


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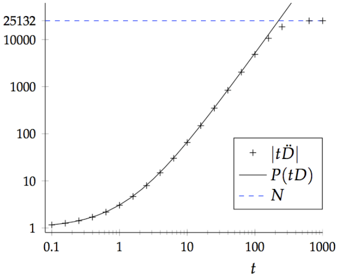


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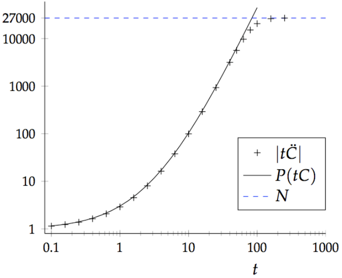


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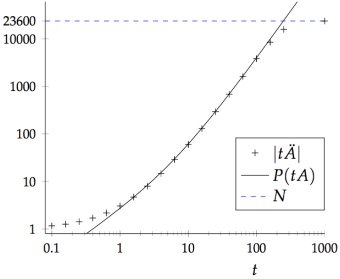

# Fractals: Ternary Cantor sets

$T_\ell^0 :=$ 

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$T_\ell^2 :=$

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$$\left| T_\ell^k \right| = 1 + \frac{1}{2} \sum_{i=1}^{k} 2^i \tanh\left(\frac{\ell}{2 \cdot 3^i}\right) + 2^k \tanh\left(\frac{\ell}{2 \cdot 3^k}\right)$$

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Lemma: Suppse $p$ is a function on $\{T_\ell\}$ then $p$ satisfies the inclusion-exclusion principle if and only if

$$p(T_\ell) = f(\ell) \cdot \ell^{\log_3 2}$$

for some $f\colon (0,\infty) \to \mathbb{R}$ with $f(3\ell) = f(\ell)$.

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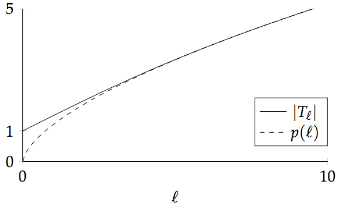

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Asymptotic Principle: There is a large class $\mathcal{C}$ of compact subsets of Euclidean space and a function $p\colon \mathcal{C} \to \mathbb{R}$ which is tractable and interesting, possibly related to valuations, such that for $A \in \mathcal{C}$

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For example

- finite sets of points [$p$ = cardinality = $P$]
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Guess: For $A$ the closure of an open set $p(A) = P(A)$.

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Theorem (Meckes): If $A \subset \mathbb{R}^m$ and $\|A\|$ exists then $\|A\| = |A|$.

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Suppose $A$ a homogeneous metric space and $\mu$ an invariant measure.
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Theorem (Meckes): $\|S_R^n\| = \left|S_R^n\right|$.

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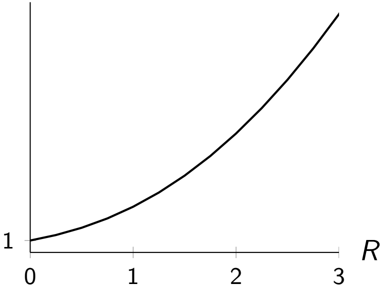

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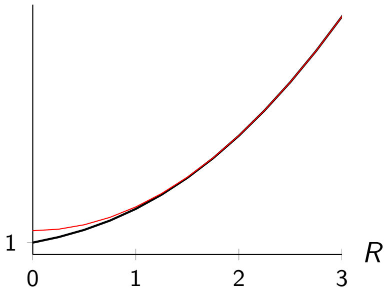

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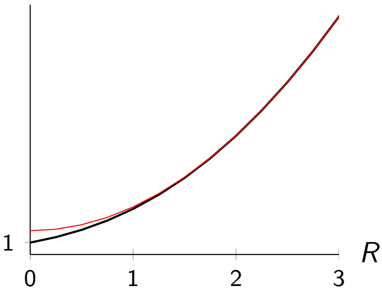
$$+\, O(R^{-1}) \qquad \text{as } R \to \infty.$$



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For example

Suppose $\Sigma$ is a homogeneous Riemannian 2-sphere or 2-torus

$$\|t\Sigma\| = \frac{\mathrm{Area}(t\Sigma)}{2\pi} + \chi(t\Sigma) + O(t^{-2}) \qquad \text{as } t \to \infty.$$