Magnitude and other measures of metric spaces

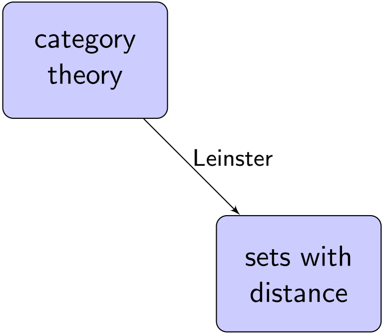
Simon Willerton
University of Sheffield

Exploratory meeting on
the mathematics of biodiversity
CRM Barcelona
July 2012

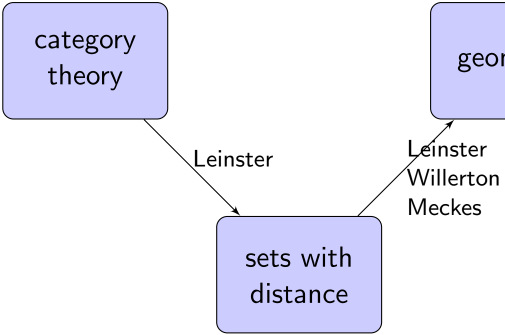
Overview

category theory

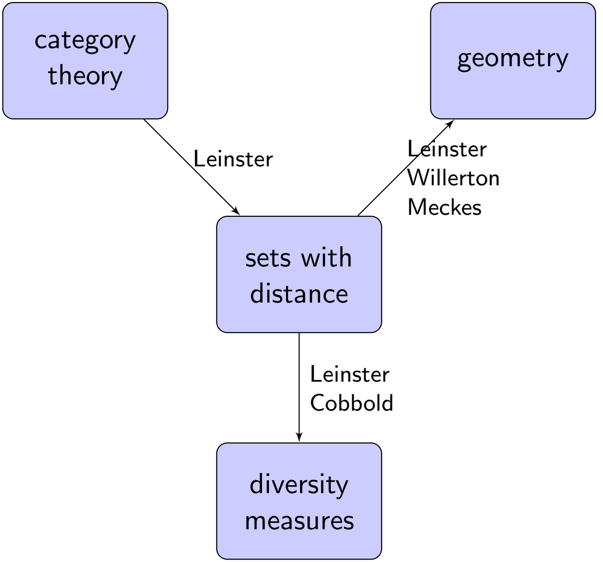
Overview



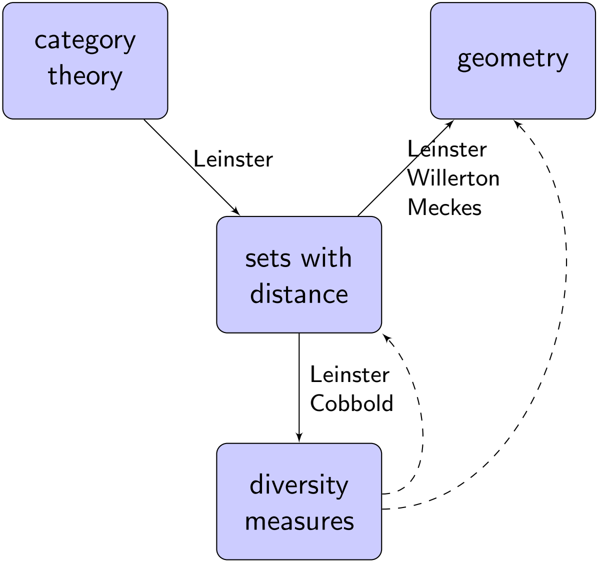
Overview


category theory
geometry
Leinster
Leinster
Willerton
Meckes
sets with distance

Overview


category theory
geometry
Leinster
Leinster
Willerton
Meckes
sets with distance
Leinster
Cobbold
diversity measures

Overview


category theory
geometry
Leinster
Leinster
Willerton
Meckes
sets with distance
Leinster
Cobbold
diversity measures

“set with distances” = “metric space”

We have

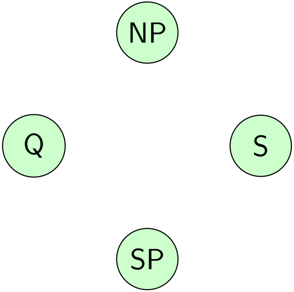
- a set of ‘points’
- some notion of distance $0 \leqslant d_{ij} \leqslant \infty$ between the ith and jth points.

"set with distances" = "metric space"

We have

- a set of 'points'
- some notion of distance $0 \leqslant d_{ij} \leqslant \infty$ between the ith and jth points.

For example:

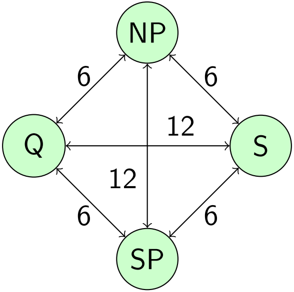


"set with distances" = "metric space"

We have

- a set of 'points'
- some notion of distance $0 \leqslant d_{ij} \leqslant \infty$ between the ith and jth points.

For example:

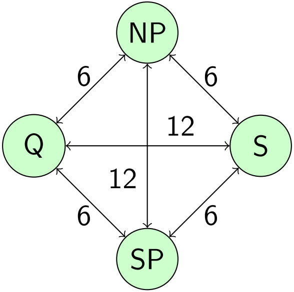


“set with distances” = “metric space”

We have

- a set of ‘points’
- some notion of distance $0 \leqslant d_{ij} \leqslant \infty$ between the ith and jth points.

For example:



Note: not every metric space can be thought of as points in Euclidean space.

Magnitude [Leinster]

Metric space X with similarity matrix $Z_{ij} := e^{-d_{ij}}$.

Magnitude [Leinster]

Metric space X with similarity matrix $Z_{ij} := e^{-d_{ij}}$.

Define 'weight' (if possible) $-\infty < w_i < \infty$ at each point i so that

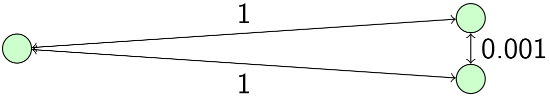
$$\sum_j Z_{ij} w_j = 1 \qquad \text{for every } j$$

Magnitude [Leinster]

Metric space X with similarity matrix $Z_{ij} := e^{-d_{ij}}$.

Define 'weight' (if possible) $-\infty < w_i < \infty$ at each point i so that

$$\sum_j Z_{ij} w_j = 1 \qquad \text{for every } j$$

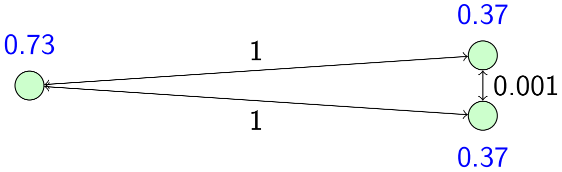


Magnitude [Leinster]

Metric space X with similarity matrix $Z_{ij} := e^{-d_{ij}}$.

Define 'weight' (if possible) $-\infty < w_i < \infty$ at each point i so that

$$\sum_j Z_{ij} w_j = 1 \qquad \text{for every } j$$

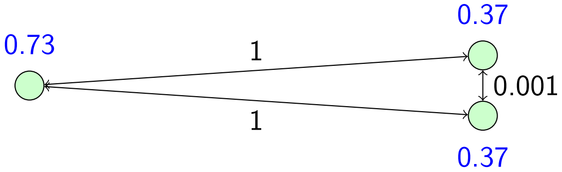


Magnitude [Leinster]

Metric space X with similarity matrix $Z_{ij} := e^{-d_{ij}}$.

Define 'weight' (if possible) $-\infty < w_i < \infty$ at each point i so that

$$\sum_j Z_{ij} w_j = 1 \qquad \text{for every } j$$



Define the magnitude by

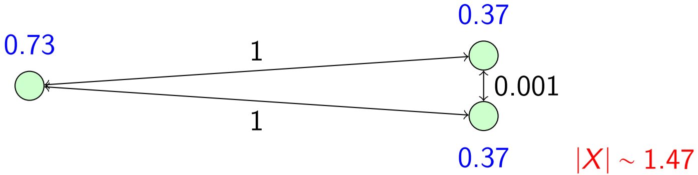
$$|X| = \sum_i w_i$$

Magnitude [Leinster]

Metric space X with similarity matrix $Z_{ij} := e^{-d_{ij}}$.

Define 'weight' (if possible) $-\infty < w_i < \infty$ at each point i so that

$$\sum_j Z_{ij} w_j = 1 \qquad \text{for every } j$$



Define the magnitude by

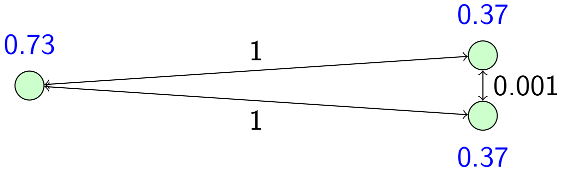
$$|X| = \sum_i w_i$$

Magnitude [Leinster]

Metric space X with similarity matrix $Z_{ij} := e^{-d_{ij}}$.

Define 'weight' (if possible) $-\infty < w_i < \infty$ at each point i so that

$$\sum_j Z_{ij} w_j = 1 \qquad \text{for every } j$$



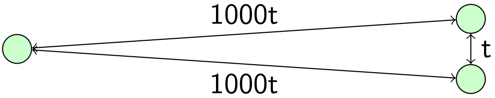
$|X| \sim 1.47$

Define the magnitude by

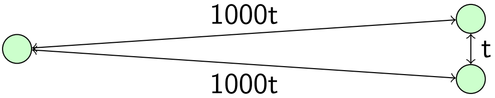
$$|X| = \sum_i w_i$$

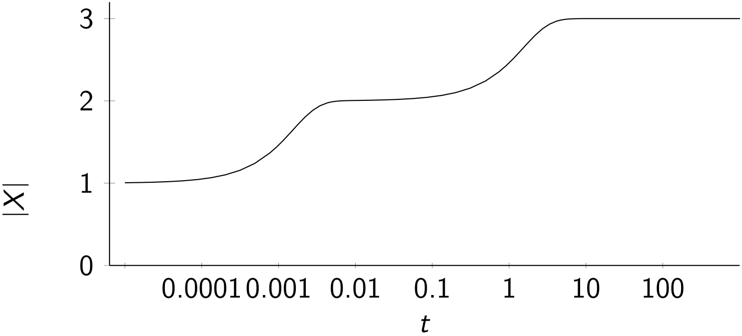
If Z_{ij} is invertible then $|X| = \sum_{ij} (Z^{-1})_{ij}$.

Example of scaling

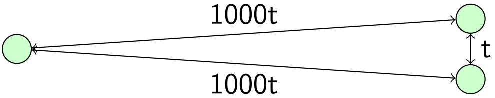


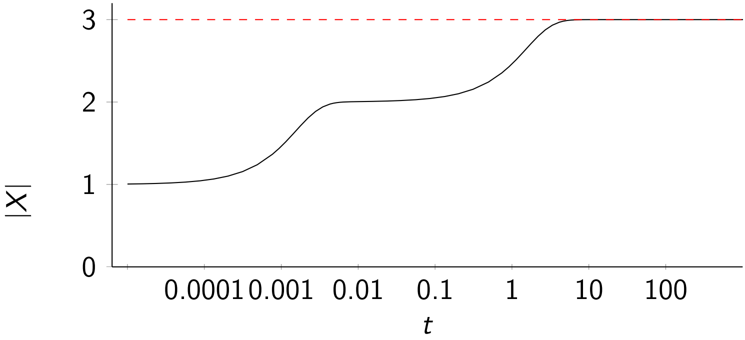
Example of scaling



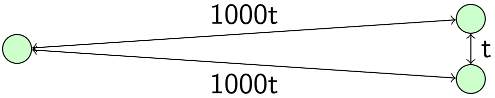


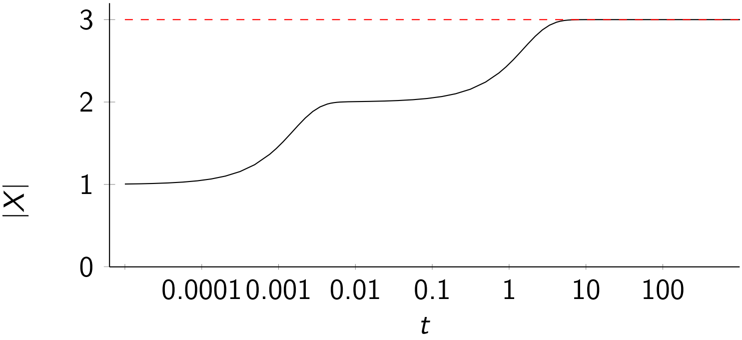
Example of scaling





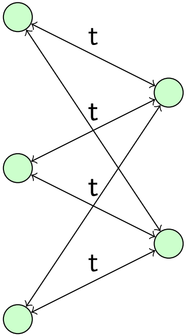
Example of scaling



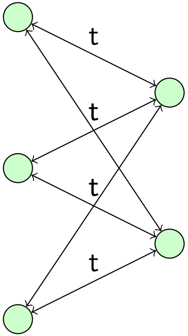


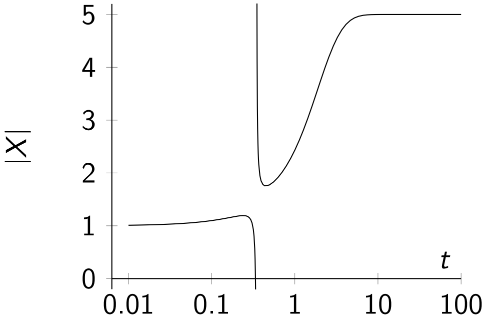
As any space X is scaled bigger and bigger $|X| \to N$.

Example of bad metric space

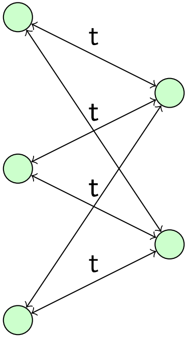


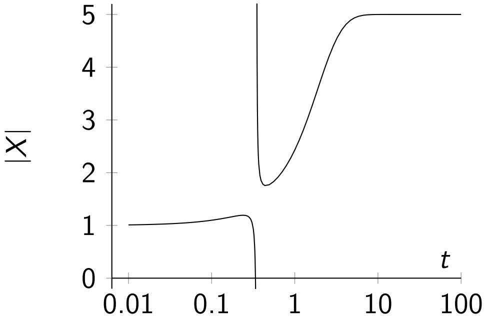
Example of bad metric space





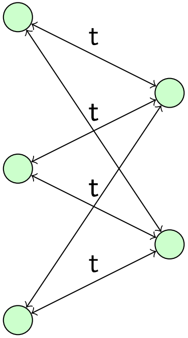
Example of bad metric space

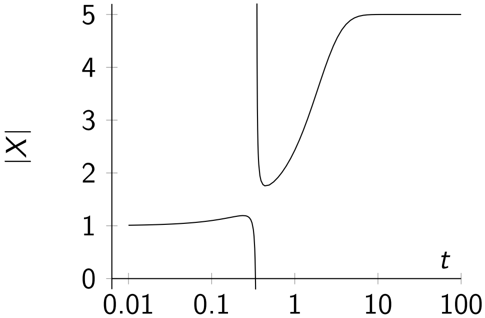




Many metric spaces are better behaved than this.

Example of bad metric space





Many metric spaces are better behaved than this.

If Z is positive definite then $|X|$ is defined.

For example, if X is a subset of Euclidean space then $|X|$ is defined.

Diversity measures [Leinster, Cobbold]

Model our community using

- a metric space X with similarity matrix Z_{ij}
- a probability (or relative abundance) p_i at the ith point.

Diversity measures [Leinster, Cobbold]

Model our community using

- a metric space X with similarity matrix Z_{ij}
- a probability (or relative abundance) p_i at the ith point.

Effective number of species:

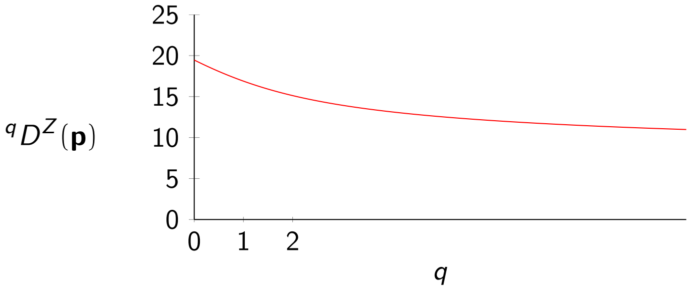
$$
{}^{q}D^{Z}(\mathbf{p}) := \begin{cases} \left(\sum\limits_{i:p_i>0} p_i (Z\mathbf{p})_i^{q-1} \right)^{\frac{1}{1-q}} & q \neq 1, \\ \prod\limits_{i:p_i>0} (Z\mathbf{p})_i^{-p_i} & q = 1, \\ \min\limits_{i:p_i>0} \dfrac{1}{(Z\mathbf{p})_i} & q = \infty. \end{cases}
$$

Diversity measures [Leinster, Cobbold]

Model our community using

- a metric space X with similarity matrix Z_{ij}
- a probability (or relative abundance) p_i at the ith point.

Effective number of species:

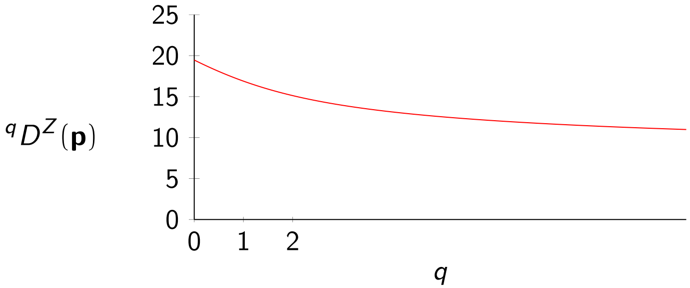


Diversity measures [Leinster, Cobbold]

Model our community using

- a metric space X with similarity matrix Z_{ij}
- a probability (or relative abundance) p_i at the ith point.

Effective number of species:



Recover various other measures of diversity using this.
For example, obtain Hill numbers when $d_{ij} = \infty$ (i.e. $Z_{ij} = 0$) for $i \neq j$.

Leinster's maximazing result

Theorem

Let X be a symmetric metric space. So Z is symmetric.

Leinster's maximazing result

Theorem

Let X be a symmetric metric space. So Z is symmetric.

- *If Z is positive definite and there is a weighting with non-negative weights ($w_i \geqslant 0$), then*

  $$D_{max}(Z) = |X|$$

  i.e., the magnitude is the maximum diversity for all q, and normalizing the weights gives the maximizing probability distribution

  $$p_i := \frac{w_i}{\sum w_i}$$

Leinster's maximazing result

Theorem

Let X be a symmetric metric space. So Z is symmetric.

- *If Z is positive definite and there is a weighting with non-negative weights ($w_i \geqslant 0$), then*

$$D_{max}(Z) = |X|$$

i.e., the magnitude is the maximum diversity for all q, and normalizing the weights gives the maximizing probability distribution

$$p_i := \frac{w_i}{\sum w_i}$$

- *Otherwise*

$$D_{max}(Z) = \max_{Y \subset X \& w_i > 0} |Y|.$$

Summary of magnitude $|X|$

- Mathematically natural (if mysterious), c.f. category theory.
- Related to biodiversity.
- Seemingly related to geometry in Euclidean space.
- Can behave rather weirdly at times.

Other size measures of metric spaces

- Get Hill Numbers by giving a probability space a dull metric.
- Get numbers for a metric space by giving a dull probability distribution.

Other size measures of metric spaces

- Get Hill Numbers by giving a probability space a dull metric.
- Get numbers for a metric space by giving a dull probability distribution.

$$ {}^{q}E(X) := {}^{q}D^{Z}\left(\left(\tfrac{1}{N}, \ldots, \tfrac{1}{N}\right)\right) $$

Other size measures of metric spaces

- Get Hill Numbers by giving a probability space a dull metric.
- Get numbers for a metric space by giving a dull probability distribution.

$$^{q}E(X) := {}^{q}D^{Z}\left(\left(\tfrac{1}{N}, \ldots, \tfrac{1}{N}\right)\right)$$

For example, analogue of species richness:

$$^{0}E(X) := \sum_{i=1}^{N}\left(\sum_{j=1}^{N} Z_{ij}\right)^{-1}$$

Other size measures of metric spaces

- Get Hill Numbers by giving a probability space a dull metric.
- Get numbers for a metric space by giving a dull probability distribution.

$$^{q}E(X) := {}^{q}D^{Z}\left(\left(\tfrac{1}{N}, \dots, \tfrac{1}{N}\right)\right)$$

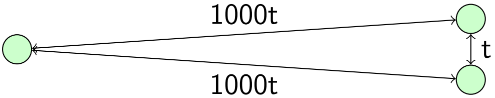
For example, analogue of species richness:

$$^{0}E(X) := \sum_{i=1}^{N}\left(\sum_{j=1}^{N} Z_{ij}\right)^{-1}$$

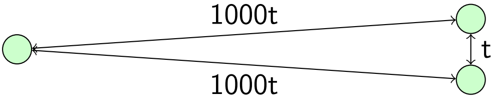
Note: this is not the same as

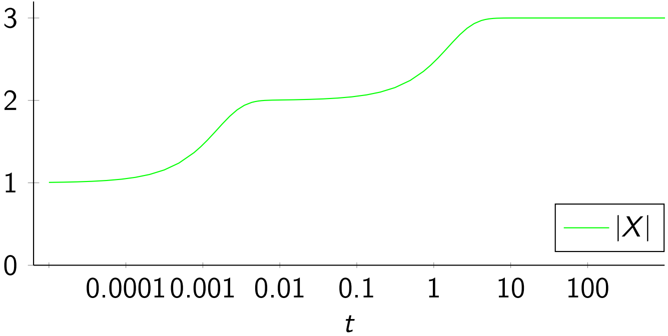
$$|X| = \sum_{i=1}^{N}\sum_{j=1}^{N}(Z)^{-1}_{ij}$$

Example of scaling II

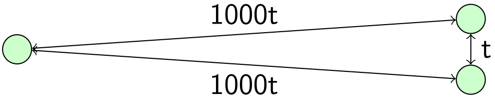


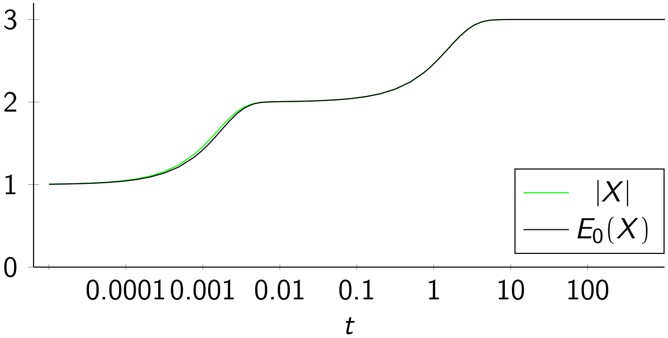
Example of scaling II



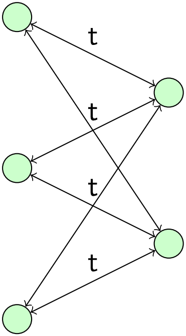


Example of scaling II

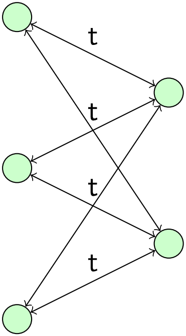


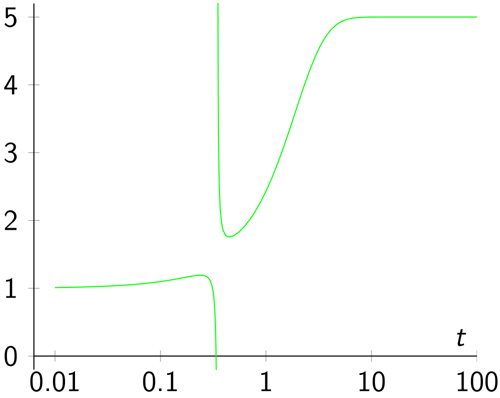


Example of bad metric space II

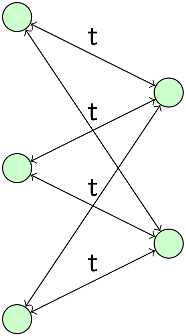


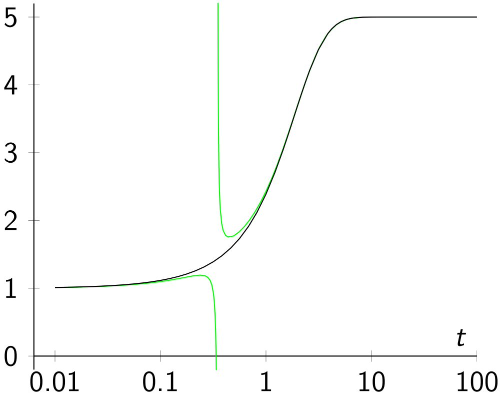
Example of bad metric space II



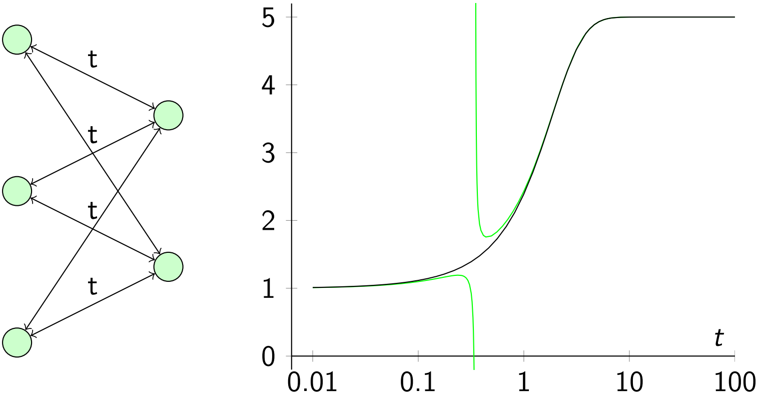


Example of bad metric space II





Example of bad metric space II



- The size ${}^0E(X)$ is defined for all metric spaces.
- As X is scaled up ${}^0E(X)$ increases from 1 to N.
- It is much easier to calculate ${}^0E(X)$ than $|X|$.

Zooming in on a space with 6400 points

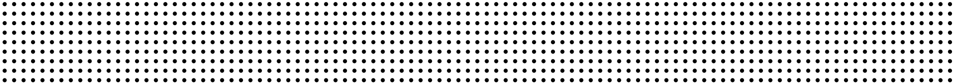
Zooming in on a space with 6400 points

Zooming in on a space with 6400 points

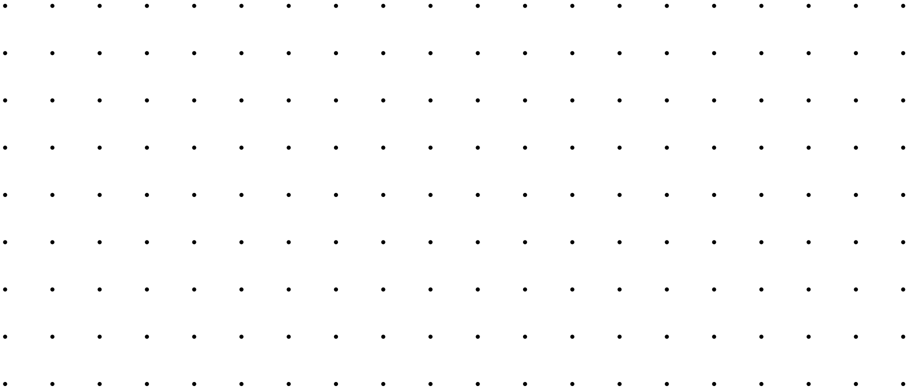
Zooming in on a space with 6400 points

Zooming in on a space with 6400 points

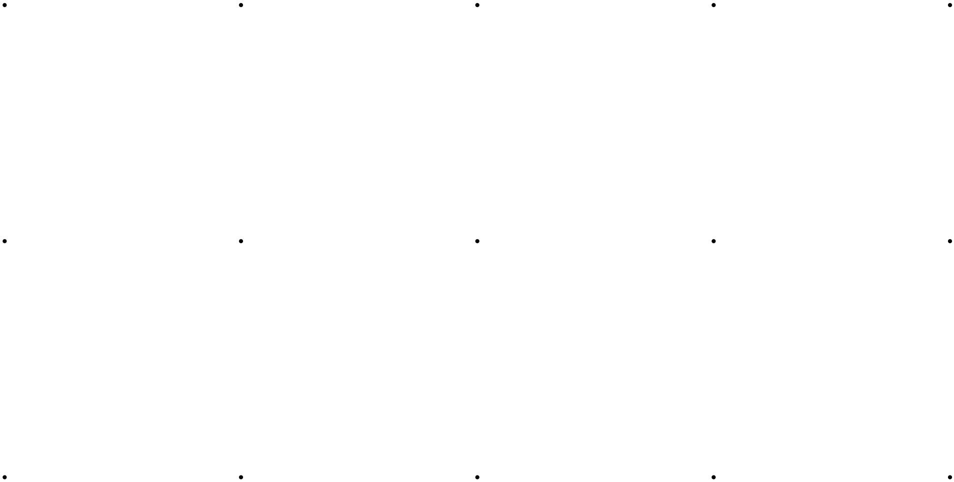
Zooming in on a space with 6400 points



Zooming in on a space with 6400 points



Zooming in on a space with 6400 points



Dimension

In a metric space we can scale all the distances.
What should happen to the size?

Dimension

In a metric space we can scale all the distances.
What should happen to the size?

For example, double the distances:

Dimension

In a metric space we can scale all the distances.
What should happen to the size?

For example, double the distances:

Dimension

In a metric space we can scale all the distances.
What should happen to the size?

For example, double the distances:

Dimension

In a metric space we can scale all the distances.
What should happen to the size?

For example, double the distances:

Dimension

In a metric space we can scale all the distances.
What should happen to the size?

For example, double the distances:

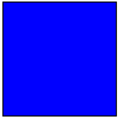
2 times as big

Dimension

In a metric space we can scale all the distances.
What should happen to the size?

For example, double the distances:

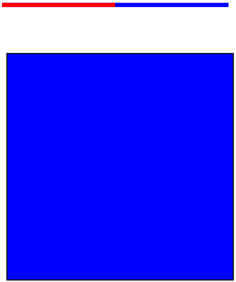
2 times as big



Dimension

In a metric space we can scale all the distances.
What should happen to the size?

For example, double the distances:



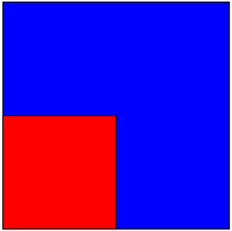
2 times as big

Dimension

In a metric space we can scale all the distances.
What should happen to the size?

For example, double the distances:

2 times as big

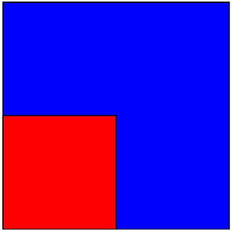


Dimension

In a metric space we can scale all the distances.
What should happen to the size?

For example, double the distances:

2 times as big



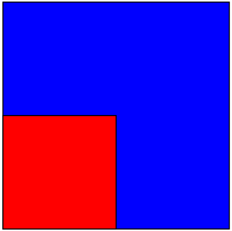
4 times as big

Dimension

In a metric space we can scale all the distances.
What should happen to the size?

For example, double the distances:

$2^1 = 2$ times as big



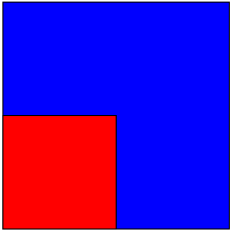
$2^2 = 4$ times as big

Dimension

In a metric space we can scale all the distances.
What should happen to the size?

For example, double the distances:

$2^1 = 2$ times as big



$2^2 = 4$ times as big

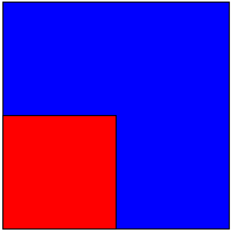
Think of dimension as how the size changes when the distances are changed.

Dimension

In a metric space we can scale all the distances.
What should happen to the size?

For example, double the distances:

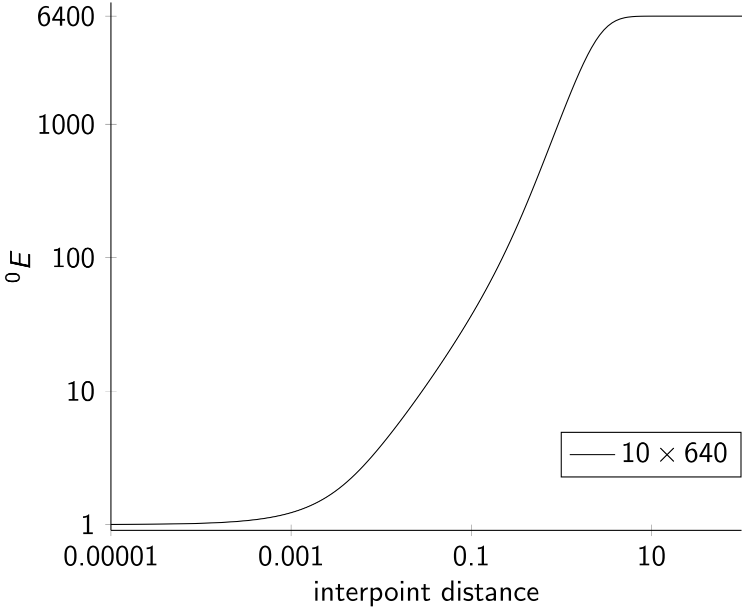
$2^1 = 2$ times as big



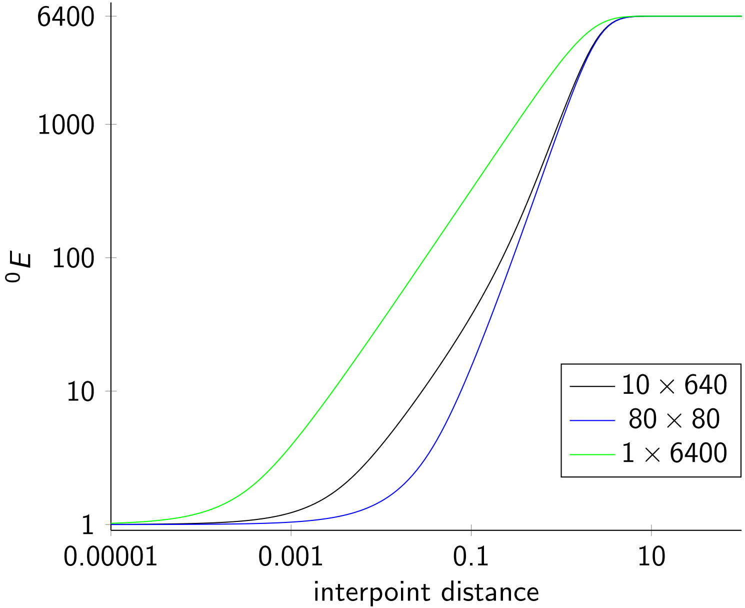
$2^2 = 4$ times as big

Think of dimension as how the size changes when the distances are changed.
Given 'size' can see if it gives a good idea of dimension.

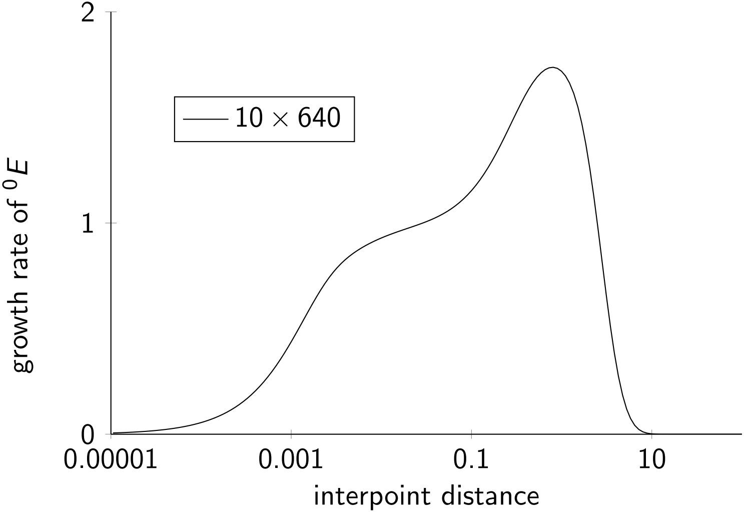
Size of rectangles with 6400 points



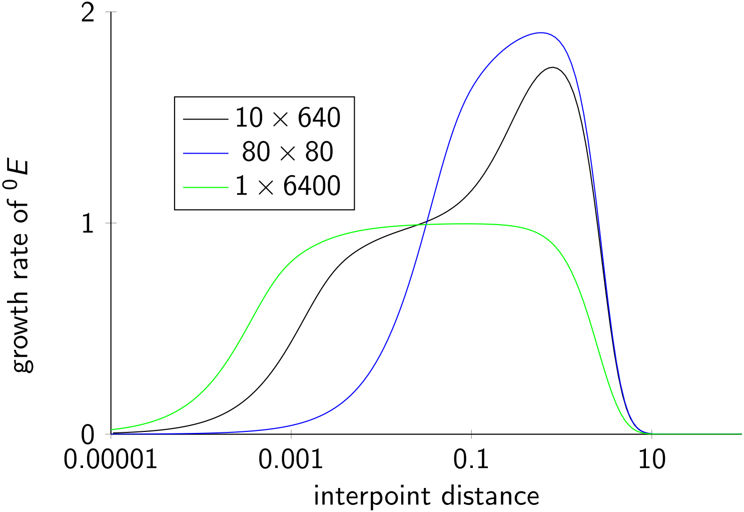
Size of rectangles with 6400 points



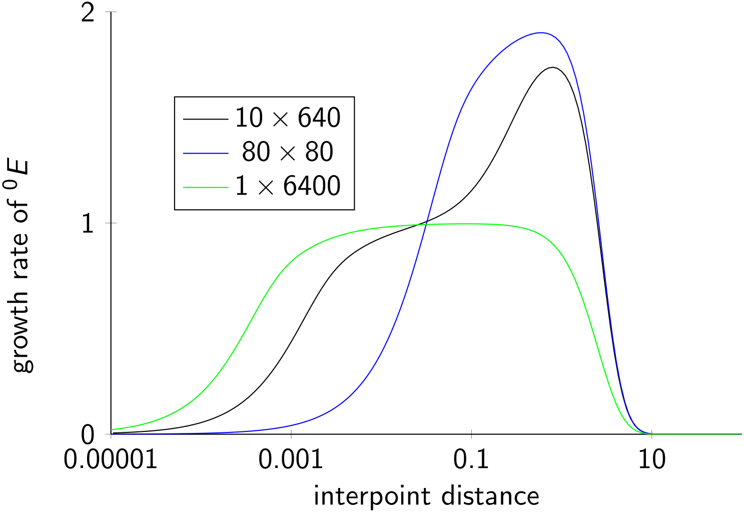
Rectangles with 6400 points and ‘dimension’



Rectangles with 6400 points and 'dimension'



Rectangles with 6400 points and ‘dimension’



There is geometric information is $^0E(X)$.