The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2016–17

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1 (i) Determine whether the sequences, whose nth terms are given below, converge or diverge. Give brief reasons in all cases, and state the limits if they exist.

$$\frac{5n^4+6n^2+n+2}{(2n^2+5n+1)^2}\,, \qquad \left(\frac{1}{3}\right)^{\frac{1}{n}}+\left(\frac{1}{3}\right)^{n}, \qquad \frac{n^{30}3^{3n}+n^3 30^n}{5^{2n}+2^{5n}}\,, \qquad (-1)^n n^{1/n}.$$

(12 marks)

(ii) Give the *formal* definition of the notion of a sequence of real numbers *converging to a limit*. Use this definition to prove that the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ converges to a limit, which you should state. Use the definition of a limit to show that if (x_n) and (y_n) are sequences with $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$. ***(13 marks)***

2 (i) Determine whether the sequences, whose nth terms are given below, converge or diverge. Give brief but clear reasons in all cases, and state the limits if they exist.

$$\frac{6n^2+n+2}{(5n+1)^2}\,, \qquad \left(\frac{1}{2}\right)^{\frac{1}{n}}+\left(\frac{1}{2}\right)^{n}, \qquad \frac{n^{30}3^{2n}+n^3 30^n}{5^{2n}+2^{5n}}\,, \qquad (-1)^n \cos(1/n).$$

(12 marks)

(ii) Give the *formal* definition of the notion of a sequence of real numbers *converging to a limit*. Use this definition to prove that the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots$ converges to a limit, which you should state.

Use the definition of a limit to show that if (x_n), (y_n) and (z_n) are sequences with $x_n \to l$, $z_n \to l$ and $z_n \leqslant y_n \leqslant x_n$ for all n then $y_n \to l$. ***(13 marks)***

3 (i) Determine whether the sequences, whose nth terms are given below, converge or diverge. Give brief reasons in all cases, and state the limits if they exist.

$$\frac{7n^2+3n}{(2n+14)^2}, \qquad \left(\frac{5n+3}{2n+1}\right)^{\frac{1}{n}}, \qquad \frac{n^{221}(2^{\frac{n}{2}}+3^{\frac{n}{3}})}{3^{3n}+4^{4n}}, \qquad 3^{\frac{1}{n}}\cdot\sin\left(\tfrac{n\pi}{2}\right).$$ ***(12 marks)***

(ii) Give the *formal* definition of the notion of a sequence of real numbers *converging to a limit*. ***(4 marks)***

Use this definition to prove that the sequence $\frac{1}{3}, \frac{4}{5}, \frac{7}{7}, \frac{10}{9}, \ldots$ converges to a limit, which you should state. Use properties of null sequences to show that if (x_n) and (y_n) are sequences with $x_n \to x$ and $y_n \to y$ then $x_n y_n \to xy$. ***(9 marks)***

4 (i) Determine whether the sequences, whose nth terms are given below, converge or diverge. Give brief reasons in all cases, and state the limits if they exist.

$$\frac{4n^2+n+2}{(4n+1)^2}, \qquad \left(\frac{1}{6}\right)^{\frac{1}{n}}+\left(\frac{1}{6}\right)^{n}, \qquad \frac{n^{35}3^{2n}+n^3 20^n}{2^{4n}+5^{2n}}, \qquad (-1)^n\left(\frac{n-1}{n}\right).$$

(12 marks)

(ii) Give the *formal* definition of the notion of a sequence of real numbers *converging to a limit*. Use this definition to prove that the sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$ converges to a limit. Use the definition of a limit to show that if (x_n) and (y_n) are sequences with $x_n \to 0$ and $y_n \to 0$ then $x_n + y_n \to 0$. ***(13 marks)***

5 (i) Determine whether the sequences, whose nth terms are given below, converge or diverge. Give brief, precise reasons in all cases, and state the limits if they exist.

$$x_n := \frac{\sqrt{3n}+7}{\sqrt{n}+5}, \qquad y_n := \begin{cases} \left(\frac{1}{6}\right)^{\frac{1}{n}} & n \text{ even} \\ \left(\frac{1}{6}\right)^{n} & n \text{ odd} \end{cases},$$

$$z_n := \frac{n^2 2^n + 3^n}{4^n+5^n}, \qquad w_n := \cos\left(\tfrac{1}{n}\right).$$

(12 marks)

(ii) Give the *formal* definition of the notion of a sequence of real numbers *converging to a limit*. Use this definition to prove that the sequence $\left(\frac{1}{\sqrt{n}}\right)$ converges to a limit. Use the definition of a limit to show that if c is a real constant and (x_n) is a sequence with $x_n \to x$ then $cx_n \to cx$.

(13 marks)

6 State which of the statements below are true and which are false. Prove those that are true, and provide counter examples for those that are false. Theorems proved in lectures may be used without proof, provided that they are precisely stated.

(a) Every set with a supremum has a maximum element.

(b) Every set with a maximum element has a supremum.

(c) Every increasing sequence of negative numbers converges.

(d) Every sequence which is bounded above is convergent.

(e) Every convergent sequence is bounded above.

(f) If the sequence (x_n) converges then $(x_{n+1} - x_n)$ converges to zero.

(g) If the sequence (x_n) is such that $(x_{n+1} - x_n)$ converges to zero then (x_n) is convergent. ***(25 marks)***

7 (i) Give the formal definition of a *decreasing sequence* and an *increasing sequence*. State and prove the "Spanish Hotels Theorem". ***(10 marks)***

(ii) By differentiating the function $f(x) := xe^{-x}$ show that the sequence $x_n := ne^{-n}$ is decreasing. Deduce that this tends to a limit, explaining your reasoning. Call this limit k. Giving a different reason in each case, explain why

$$2ne^{-2n} \to k \qquad \text{and} \qquad n^2e^{-2n} \to k^2.$$

By looking at the ratio of these two sequences, prove that $k = 0$. ***(15 marks)***

8 State which of the statements below are true and which are false. Prove those that are true, and provide counter examples for those that are false. Theorems proved in lectures may be used without proof, provided that they are precisely stated.

(a) Every real number is the supremum of a set of rational numbers. ***(4 marks)***

(b) Every set with a maximum element has a supremum. ***(4 marks)***

(c) Every increasing sequence of negative numbers converges. ***(4 marks)***

(d) Every sequence which is convergent is bounded above. ***(4 marks)***

(e) Every convergent sequence has an increasing subsequence. ***(4 marks)***

(f) Every non-empty set of rational numbers with a rational upper bound has a rational supremum. ***(5 marks)***

9 State which of the statements below are true and which are false. Prove those that are true, and provide counter examples for those that are false. Theorems proved in lectures may be used without proof, provided that they are precisely stated.

(a) Every set with a maximum and a minimum element is finite.

(b) Every non-empty finite set has a maximum and a minimum element.

(c) Every set of rational numbers with a rational lower bound has a rational infimum.

(d) No set has exactly one lower bound.

(e) Every bounded sequence is convergent.

(f) If a convergent sequence has an infinite number of positive terms and an infinite number of negative terms then it converges to zero. ***(25 marks)***

10 State which of the statements below are true and which are false. Prove those that are true, and provide counter examples with an explanation for those that are false. Theorems proved in lectures may be used without proof, provided that they are precisely stated.

(a) Every set with a minimum element is bounded below.

(b) Every set which is bounded below has a minimum element.

(c) Every sequence which is bounded above has a convergent subsequence.

(d) An increasing sequence cannot contain a strictly decreasing subsequence.

(e) A convergent sequence with a strictly increasing subsequence cannot contain a strictly decreasing subsequence.

(f) If E is a set of rational numbers, with a maximum but no minimum, then the difference $\sup E - \inf E$ must be rational. ***(25 marks)***

11 Consider sequences $x_1, x_2, x_3, \ldots$ satisfying the recurrence relation

$$x_{n+1} = \frac{1}{10}\left(x_n^2 + 21\right).$$

By expressing $x_{n+1} - x_n$ as a factorized quadratic, show that if such a sequence converges, then it must be to one of *two* possible values. ***(6 marks)***

Consider the case when $3 \leqslant x_1 < 7$. Prove that $3 \leqslant x_n < 7$ for all n. Use your expression for $x_{n+1} - x_n$ to prove that $x_1, x_2, x_3, \ldots$ decreases. Deduce that it converges, and determine its limit. ***(10 marks)***

Consider next the case when $x_1 > 7$. Show that $x_n > 7$ for all n, and that $x_1, x_2, x_3, \ldots$ increases. What can you conclude about the limiting behaviour of this sequence? Give reasons for your answer. ***(6 marks)***

12 (i) Give the formal definition of what it means for a real-valued function f to be *continuous* at the point a in its domain.
Give an example of a function $j\colon \mathbb{R} \to \mathbb{R}$ which is continuous at $x = 0$. Sketch its graph, illustrating which feature corresponds to it being continuous. Prove that it is continuous.
If $f, g\colon \mathbb{R} \to \mathbb{R}$ are continuous functions, prove that the composition $f \circ g$, defined by $f \circ g(x) := f(g(x))$, is also continuous. ***(13 marks)***

(ii) If f is a real-valued function, give the definition of the *derivative* of f. Use the definition to find the derivative of $\sin(x)$ and $\cos(x)$. You may use the fact that $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$.
Give an example of a continuous function $k\colon \mathbb{R} \to \mathbb{R}$ which is not differentiable at $x = 2$. Prove that your function is not differentiable at $x = 2$. ***(12 marks)***

13 (i) Give a formal definition of what it means for a real valued function $f(x)$ to be *continuous* at the point a of its domain. ***(3 marks)***

Prove that if $f(x)$ and $g(x)$ are real valued functions, both continuous at a, then $f(x)g(x)$ is also continuous at a.

Give an example to show that $f(x)g(x)$ can be continuous even if $f(x)$ is not continuous. ***(7 marks)***

(ii) If $f(x)$ is a real-valued function, give the definition of its *derivative*. ***(4 marks)***

Use the definition to find the derivative of $f(x) = \sin(x)$.

Show from first principles that if $g(x)$ is differentiable and non-zero then $\dfrac{1}{g(x)}$ is also differentiable, and find the derivative. Hence find the derivative of $h(x) := \csc(x)$.

Prove that the function $k(x) := |x|$ is not differentiable at $x = 0$. ***(11 marks)***

(iii) For each of the following, give an example of such a function and sketch its graph:

(a) a continuous function $f\colon (0, 1) \to \mathbb{R}$ which is not bounded;

(b) a continuous function $g\colon (0, 1) \to \mathbb{R}$ which is bounded but has no maximum;

(c) a function $h\colon [0, 1] \to \mathbb{R}$ which is not bounded;

(d) a function $k\colon [0, 1] \to \mathbb{R}$ which is bounded but has no maximum.

State a theorem which guarantees that certain functions defined on an interval have a maximum and a minimum. Why is it that none of the above examples you have given contradicts this theorem? ***(15 marks)***

(iv) If f is a real-valued function, give the definition of the *derivative* of f. Use the definition to find the derivative of $f(x) := x^n$ where n is a positive integer.
Prove that the function $k(x) := |x|$ is not differentiable at $x = 0$. ***(10 marks)***

14 (i) Define what it means for a real-valued function to be *continuous*. For each of the following, give an example of such a function and sketch its graph (you do not need to prove that it has the required property):

(a) an *unbounded* continuous function $f\colon [-1,0) \cup (0,1] \to \mathbb{R}$ which does *not* extend to a continuous function on $[-1,1]$;

(b) a *bounded* continuous function $g\colon [-1,0) \cup (0,1] \to \mathbb{R}$ which does *not* extend to a continuous function on $[-1,1]$;

(c) a differentiable function $h\colon [-1,0) \cup (0,1] \to \mathbb{R}$ which extends to a continuous function on $[-1,1]$ but does not extend to a differentiable one. ***(13 marks)***

(ii) State Rolle's Theorem.

Suppose that $k\colon [a,b] \to \mathbb{R}$ and $l\colon [a,b] \to \mathbb{R}$ are continuous functions which are differentiable on (a,b), such that $k(a) \neq k(b)$ and such that there is *no* $t \in (a,b)$ with $k'(t) = l'(t) = 0$. Show that there is a $c \in (a,b)$ such that

$$\frac{l(b) - l(a)}{k(b) - k(a)} = \frac{l'(c)}{k'(c)}.$$

Hint: consider the function $h(t) := \big(l(b) - l(a)\big)k(t) - \big(k(b) - k(a)\big)l(t)$.

For k and l as above, consider the curve given by the parametric equations $x = k(t)$, $y = l(t)$ for $t \in [a,b]$. If $A = \big(k(a), l(a)\big)$ and $B = \big(k(b), l(b)\big)$ are the two end points of the curve then show that there is a point on the curve whose tangent line is parallel to the line AB.

(12 marks)

15 (i) State the *Intermediate Value Theorem* and use it to show that the equation $2x^2(x+2)-1 = 0$ has a root in each of the intervals $(-2,-1)$, $(-1,0)$ and $(0,1)$. Consider the root in $(0,1)$; use three iterations of the bisection method to find a smaller interval in which this root lies. ***(10 marks)***

(ii) State *Rolle's Theorem* and the *Mean Value Theorem*. Indicate how the latter can be deduced from the former. ***(10 marks)***

Apply the Mean Value Theorem to the function $f(x) = \ln x$ between 1 and b to deduce that for $b > 1$

$$\frac{b-1}{b} < \ln b < b - 1.$$

(5 marks)

16 (i) State the *Intermediate Value Theorem*. Prove that the equation

$$x \cos x = \sin x$$

has a solution between $2m\pi$ and $(2m+1)\pi$ for every positive integer m. By considering $\frac{d}{dx}(x \cos x - \sin x)$, prove that there is only one solution in each of these intervals.

(13 marks)

(ii) State the *Mean Value Theorem*.

Apply the Mean Value Theorem to the function e^x on the interval $[0, b]$, where $0 < b$, to show that

$$b < e^b - 1 < be^b.$$

(12 marks)

17 (i) State the *Intermediate Value Theorem* (IVT). ***(4 marks)***

Use the IVT to show that the equation $x^3 + 6x^2 + 3x - 4 = 0$ has a root in each of the intervals $(-6, -2)$, $(-2, 0)$ and $(0, 1)$. Consider the root in $(0, 1)$; use two iterations of the bisection method to find a smaller interval in which this root lies. ***(9 marks)***

(ii) State the *Mean Value Theorem* (MVT). ***(4 marks)***

Apply the Mean Value Theorem to the function $f(x) = \cos^{-1}(x)$ on the interval $[0, b]$, where $0 < b < 1$, to show that

$$\frac{\pi}{2} - b > \cos^{-1}(b) > \frac{\pi}{2} - \frac{b}{\sqrt{1-b^2}}.$$

[**Hint**: You may use the fact that $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$.]

(8 marks)

18 (i) State the *Intermediate Value Theorem* (IVT) and use it to show that the equation $2x^2(x+4)-1=0$ has a root in each of the intervals $(-4,-3)$, $(-1,0)$ and $(0,1)$, stating carefully how the IVT is used. Consider the root in $(0,1)$; use two iterations of the bisection method to find a smaller interval in which this root lies. ***(10 marks)***

(ii) Let $f:[a,b]\to\mathbb{R}$ be a continuous function that is differentiable on (a,b). Let $g:[a,b]\to\mathbb{R}$ be defined by the equation

$$g(x) = f(x) - \alpha x,$$

where α is the *unique* real number determined by the condition $g(a)=g(b)$. Express α in terms of a, b, $f(a)$ and $f(b)$. State the Mean Value Theorem and show how it can be proved by applying Rolle's Theorem to the function g. ***(10 marks)***

Apply the Mean Value Theorem to the function $f(x)=\tan^{-1}x$ on the interval $[0,b]$, where $0<b$, to show that

$$\frac{b}{1+b^2} < \tan^{-1} b < b.$$

[**Hint**: You may use the fact that $\dfrac{d}{dx}(\tan^{-1}x) = \dfrac{1}{1+x^2}$.] ***(5 marks)***

19 (i) If f is a real-valued function, give the definition of the *derivative* of f. Use the definition to find the derivative of $f(x):=x^n$ where n is a non-negative integer. ***(7 marks)***

(ii) Show from first principles that if f is a non-zero, differentiable function and g is defined by $g(x):=\dfrac{1}{f(x)}$ then g is also differentiable; give a formula for the derivative $g'(x)$. ***(5 marks)***

(iii) Use the first two parts of the question to deduce the derivative of $g(x)=x^n$ where n is a *negative* integer. ***(4 marks)***

(iv) For a positive integer m, let h the real-valued function with the maximum possible domain given by $h(x):=x^{1/m}$. What is the domain of h? State the formula for the derivative of the inverse of a function. Use this to find the derivative of h. ***(7 marks)***

(v) Indicate in one or two sentences how you would go on to obtain the formula

$$\frac{d}{dx}(x^p) = px^{p-1}$$

where p is any *rational* number. ***(2 marks)***

20 (i) Evaluate the integral $\int_0^t e^x dx$ using calculus. *(3 marks)*

Now let s_n be the lower sum for the area under the curve $y = e^x$ between $x = 0$ and $x = t$ obtained by partitioning the interval $[0, t]$ into n equally sized subintervals. Sketch a graph illustrating this. Write down an expression for s_n. Using the formula for a geometric progression, or otherwise, rewrite your expression for s_n and evaluate the limit of s_n as n tends to infinity. You may use the fact that $\frac{n}{t}(e^{t/n} - 1) \to 1$. *(11 marks)*

[Hint: The formula for a geometric progression is $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$.]
Compare this answer with the integral evaluated above. Comment. *(3 marks)*

(ii) Let $f\colon [a, b] \to \mathbb{R}$ be a function and let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of the interval $[a, b]$. Give the definition of the *upper sum* $U_P(f)$ of f with respect to the partition P, and similarly give the definition of the *lower sum* $L_P(f)$. State what it means for a function to be *Riemann integrable*.
Give, without proof, an example of a large class of functions which are Riemann integrable.
Give, without proof, an example of a function $g\colon [0, 1] \to \mathbb{R}$ which is not Riemann integrable. *(8 marks)*

21 Consider the function $f(x) := e^x$, and the partition $P_N := \{0 = x_0 < \cdots < x_N = b\}$ of the interval $[0, b]$ into N equally sized pieces, so $x_i = \frac{ib}{N}$. Define what is meant by the *upper sum* $U_{P_N}(f)$, and the *lower sum* $L_{P_N}(f)$. Draw a picture to illustrate the upper sum, and find an explicit formula for it.

Calculate the limit $\lim\limits_{N\to\infty} U_{P_N}(f)$. Comment on your answer. [You may use without proof the formula for the sum of a geometric series and the fact that $\lim\limits_{t\to 0}(e^t - 1)/t$ exists and equals 1.] *(15 marks)*

For a function g define the *upper and lower integrals* $U(g)$ and $L(g)$ over some interval I and state a general relationship between them. Say what it means for g to be *Riemann integrable*.

Give an example of a function which is not Riemann integrable. Justify your answer.

State a general theorem showing that the function $f(x) := e^x$ is integrable on $[0, b]$. *(10 marks)*

22 (i) Consider the function $f(x) := b^2 - x^2$, and the partition $P_N := \{0 = x_0 < \cdots < x_N = b\}$ of the interval $[0, b]$ into N equally sized pieces, so $x_i = \frac{ib}{N}$. Define what is meant by the *lower sum* $L_{P_N}(f)$. Draw a picture to illustrate the lower sum, and find an explicit formula for it. Calculate the limit $\lim_{N\to\infty} L_{P_N}(f)$.

[**Hint**: You may use the formula $\displaystyle\sum_{i=1}^{N} i^2 = \frac{1}{6}N(N+1)(2N+1)$.] ***(14 marks)***

(ii) Prove that if $f(x)$ is a *decreasing* function on the interval $[a, b]$ and P is a partition of $[a, b]$ into N equal pieces then

$$U_{P_N} - L_{P_N} = \frac{1}{N}(b-a)(f(a) - f(b)).$$

Use this to deduce that such a decreasing function is Riemann integrable on the interval $[a, b]$. ***(8 marks)***

(iii) Hence, or otherwise, prove that the function in part (i) is Riemann integrable on $[0, b]$ and state the value of the integral. ***(3 marks)***

23 [0910] Fix a real number $b > 0$. Consider the function $f\colon [0, b] \to \mathbb{R}$ defined by $f(x) := b - x$, and the partition $P_N := \{0 = x_0 < \cdots < x_N = b\}$ of the interval $[0, b]$ into N equally sized pieces, so $x_i = \frac{ib}{N}$.

(i) Define what is meant by the *lower sum* $L_{P_N}(f)$. Draw a picture to illustrate the lower sum, indicating the relevant features. Find an explicit formula for the lower sum, simplifying your answer as much as possible.

Hint: You may use the formula $\sum_{i=1}^{N} i = \frac{1}{2}N(N+1)$. ***(12 marks)***

(ii) Define what it means for a real-valued function to be Riemann integrable. ***(3 marks)***

(iii) Prove that the function f is Riemann integrable on $[0, b]$, and give the value of the Riemann integral. ***(7 marks)***

(iv) Explain how the answer is in agreement with elementary geometry calculation for the area under the graph of the function. ***(3 marks)***

End of Question Paper