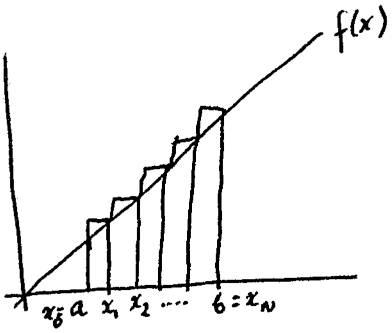
69



$f(x)$ is increasing so

$M_i = \{\sup f(x) \mid x \in [x_i, x_{i-1}]\} = f(x_i)$

$$U_{P_N}(f) = \sum_{i=1}^{N} (x_i - x_{i-1}) M_i = \sum (x_i - x_{i-1}) f(x_i)$$

$$= \sum_{i=1}^{N} \frac{(b-a)}{N}\left(a + i\frac{(b-a)}{N}\right)$$

$$= \frac{(b-a)}{N}\left\{\sum_{i=1}^{N} a + \frac{(b-a)}{N}\sum_{i=1}^{N} i\right\}$$

$$= \frac{b-a}{N}\left\{Na + \frac{b-a}{N}\frac{(N+1)N}{2}\right\}$$

$$= (b-a)\left\{a + (b-a)\left(\frac{N+1}{2N}\right)\right\} = (b-a)\left\{a + (b-a)\left(\frac{1}{2} + \frac{1}{2N}\right)\right\}$$

$$= (b-a)\left\{\tfrac{1}{2}(a+b) + \frac{(b-a)}{2N}\right\} = \underline{\underline{\frac{b^2-a^2}{2} + \frac{(b-a)^2}{2N}}}$$

Thus $\lim_{N\to\infty} U_{P_N} = \underline{\frac{b^2-a^2}{2}}$. As the upper integral $U(f)$ is the greatest lower bound of all ~~__~~ upper sums, we must have

$$\underline{U(f) \leq \frac{b^2-a^2}{2}}.$$

By a similar calculation the corresponding lower sum is given by

$$\underline{L_{P_N}(f) = \frac{b^2-a^2}{2} - \frac{(b-a)^2}{2N}}$$

So $\lim_{N\to\infty} L_{P_N}(f) = \frac{b^2-a^2}{2}$, whence we get the bound

$$\underline{L(f) \geq \frac{b^2-a^2}{2}}$$

But $U(f) \geq L(f)$ so $\frac{b^2-a^2}{2} \geq U(f) \geq L(f) \geq \frac{b^2-a^2}{2}$

and we see $U(f) = L(f)$ so f is integrable with integral $\frac{b^2-a^2}{2}$.

70 (a) A standard multiple angle formula is

$$\sin(A)\sin(B) = \tfrac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

so

$$\sin(r\alpha)\sin(\tfrac{1}{2}\alpha) = \tfrac{1}{2}\left(\cos((r-\tfrac{1}{2})\alpha) - \cos((r+\tfrac{1}{2})\alpha)\right).$$

Now $\sin(\frac{1}{2}\alpha) = 0$ if and only if $\frac{1}{2}\alpha = n\pi$ for some integer n, ie. if and only if α is an integer multiple of 2π, so provided this is not the case we have

$$\sin(r\alpha) = \frac{\cos((r-\frac{1}{2})\alpha) - \cos((r+\frac{1}{2})\alpha)}{2\sin(\frac{1}{2}\alpha)}.$$

Thus

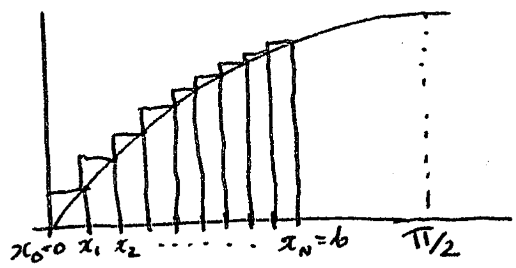
$$\sum_{r=1}^{N} \sin(r\alpha) = \sum_{r=1}^{N} \frac{\cos((r-\frac{1}{2})\alpha) - \cos((r+\frac{1}{2})\alpha)}{2\sin(\frac{1}{2}\alpha)}$$

$$= \frac{1}{2\sin(\frac{\alpha}{2})}\Big[\left(\cos(\tfrac{1}{2}\alpha) - \cancel{\cos(\tfrac{3}{2}\alpha)}\right) + \left(\cancel{\cos(\tfrac{3}{2}\alpha)} - \cancel{\cos(\tfrac{5}{2}\alpha)}\right) + \left(\cancel{\cos(\tfrac{5}{2}\alpha)} - \cancel{\cos(\tfrac{7}{2}\alpha)}\right) + \dots + \left(\cancel{\cos((N-\tfrac{1}{2})\alpha)} - \cos((N+\tfrac{1}{2})\alpha)\right)\Big]$$

$$= \frac{1}{2\sin(\frac{\alpha}{2})}\left[\cos(\tfrac{1}{2}\alpha) - \cos((N+\tfrac{1}{2})\alpha)\right]$$

as required.

(b)



Note sin is increasing on $[0,b]$, so $M_i = \sup\{f(x) \mid x \in [x_i, x_{i-1}]\}$
$= \sin(x_i) = \sin\left(\frac{ib}{N}\right)$

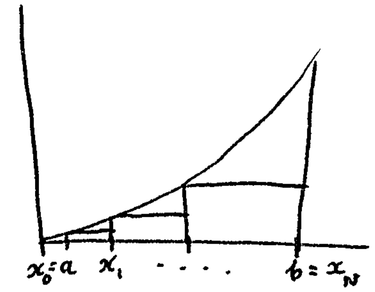
$$U_{P_N}(\sin) = \sum_{i=1}^{N}(x_i - x_{i-1})M_i = \sum_{i=1}^{N}\frac{b}{N}\sin\left(\frac{ib}{N}\right)$$

$$= \frac{b}{N}\sum_{i=1}^{N}\sin\left(i\left(\frac{b}{N}\right)\right) \underset{\text{by (a)}}{=} \frac{b}{N}\left[\frac{\cos(\frac{b}{2N}) - \cos((N+\frac{1}{2})\frac{b}{N})}{2\sin(\frac{b}{2N})}\right]$$

$$= \frac{\cos(\frac{b}{2N}) - \cos(b + \frac{b}{2N})}{\sin(\frac{b}{2N})/(b/2N)} \to \frac{\cos(0) - \cos(b)}{\underset{\text{by hint.}}{1}} \text{ as } N \to \infty$$

So $\underline{\lim_{N\to\infty} U_{P_N}(\sin) = 1 - \cos(b).}$ ~~Thus~~ This is the same as the answer we get using $\int_0^b \sin x\,dx = [-\cos x]_0^b$.

71



$f(x)=x^k \quad k>0$ increasing on $[a,b]$.

So $m_i = \inf\{f(x) \mid x\in[x_{i-1},x_i]\} = f(x_{i-1})$
$= (aq^{i-1})^k$

$$L_{P_N}(f) = \sum_{i=1}^{N}(x_i-x_{i-1})m_i = \sum_{i=1}^{N}(aq^i-aq^{i-1})(aq^{i-1})^k .$$

$$= \sum_{i=1}^{N} a(q-1)q^{i-1}a^k(q^{i-1})^k = a^{k+1}(q-1)\sum_{i=1}^{N}(q^{i-1})^{k+1}$$

$$= a^{k+1}(q-1)\sum_{i=1}^{N}(q^{k+1})^{i-1}$$

Formula for geometric progression: $\displaystyle\sum_{i=1}^{N} r^{i-1} = \frac{r^N-1}{r-1}$

So $$L_{P_N}(f) = a^{k+1}(q-1)\,\frac{(q^{k+1})^N-1}{q^{k+1}-1} = a^{k+1}\frac{q-1}{q^{k+1}-1}\left((q^N)^{k+1}-1\right)$$

$$= \frac{(q-1)}{q^{k+1}-1}\,a^{k+1}\left(\frac{b^{k+1}}{a^{k+1}}-1\right) = \frac{q-1}{q^{k+1}-1}\left(b^{k+1}-a^{k+1}\right)$$

As $f(x)$ is increasing on $[a,b]$, $M_i = \sup\{f(x)\mid x\in[x_{i-1},x_i]\} = f(x_i) = (aq^i)^k$

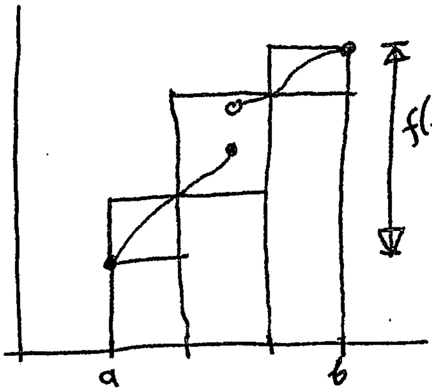
So $$U_{P_N}(f) = \sum_{i=1}^{N}(x_i-x_{i-1})M_i = \sum_{i=1}^{N}\cancel{aq^i-a}\,(x_i-x_{i-1})(aq^i)^k$$

$$= \sum_{i=1}^{N}(x_i-x_{i-1})\underbrace{(aq^{i-1})^k}_{=m_i}q^k = q^k\sum_{i=1}^{N}(x_i-x_{i-1})m_i$$

$$= q^k L_{P_N}(f).$$

As $N\to\infty$, $q\to 1$, so $q^k\to 1$, whence $\lim_{N\to\infty} U_{P_N}(f) = \lim_{N\to\infty} L_{P_N}(f)$.
So the upper and lower integrals must also be equal to this common limit. Thus $f(x)$ is integrable with $\displaystyle\int_a^b x^k\,dx = \lim_{N\to\infty} L_{P_N}(f) = \frac{1}{k+1}\left(b^{k+1}-a^{k+1}\right)$

72



As f is increasing $M_i = f(x_i)$
& $m_i = f(x_{i-1})$;

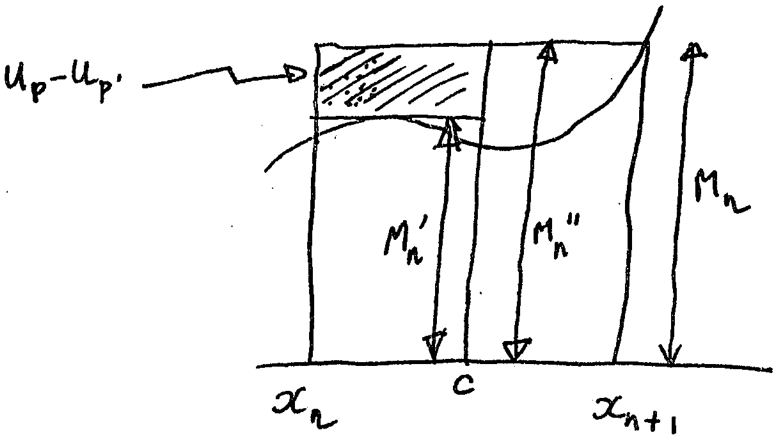
as P is equally spaced
$x_i - x_{i-1} = \frac{b-a}{N}$.

Thus

$$
\begin{aligned}
U_p(f) - L_p(f) &= \sum_{i=1}^{N} (x_i - x_{i-1}) M_i - \sum_{i=1}^{N} (x_i - x_{i-1}) m_i \\
&= \sum_{i=1}^{N} \frac{b-a}{N} f(x_i) - \sum_{i=1}^{N} \frac{b-a}{N} f(x_{i-1}) \\
&= \frac{b-a}{N} \left(\sum_{i=1}^{N} f(x_i) - \sum_{i=1}^{N} f(x_{i-1}) \right) \\
&= \frac{b-a}{N} \left(\left(\cancel{f(x_1)} + \cancel{f(x_2)} + \dots + f(x_N) \right) - \left(f(x_0) + \cancel{f(x_1)} + \dots + \cancel{f(x_{N-1})} \right) \right) \\
&= \frac{b-a}{N} \left(f(x_N) - f(x_0) \right) = \underline{\frac{b-a}{N} \left(f(b) - f(a) \right)}.
\end{aligned}
$$

Looking at the above picture you can see the difference between the upper and lower sums is represented by the top rectangles, each of which has width $\frac{b-a}{N}$, so if you slide them all under one another you get a big rectangle of width $\frac{b-a}{N}$ and height $f(b)-f(a)$, thus of area $\frac{b-a}{N}(f(b)-f(a))$.

$\underset{\sim}{73}$ Suppose $P = \{a = x_0 < \ldots < x_\nu = b\}$ and $P' = \{a = x_0 < \ldots < x_n < c < x_{n+1} < \ldots x_\nu\}$



$M_n = \sup_{x \in [x_n, x_{n+1}]} f(x)$

$M_n' = \sup_{x \in [x_n, c]} f(x)$

$M_n'' = \sup_{x \in [c, x_{n+1}]} f(x)$

The upper sums will agree everywhere except where pictured.

So
$$U_p(f) - U_{p'}(f) = (x_{n+1} - x_n) M_n - (x_{n+1} - c) M_n'' - (c - x_n) M_n'$$

But either $M_n = M_n'$ or $M_n = M_n''$

so
$$U_p(f) - U_{p'}(f) = \begin{cases} (c - x_n)(M_n - M_n') & \text{if } M_n = M_n'' \\ (x_{n+1} - c)(M_n - M_n'') & \text{if } M_n = M_n' \end{cases}$$

[The shaded box above corresponds to the first case.]

However, $(c - x_n) < \delta$ and $(x_{n+1} - c) < \delta$,

$$M_n < M, \quad M_n' > m, \quad M_n'' > m$$

So
$$U_p(f) - U_{p'}(f) < \delta(M - m)$$

as required.

The second results follows by induction.

74 If $P = \{a = x_0 < \dots < x_N = b\}$ is a partition of $[a,b]$ and $c_i \in [x_{i-1}, x_i]$ then we have

$$\sum_{i=1}^{N} (f(c_i) + g(c_i))(x_i - x_{i-1})$$
$$= \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) + \sum_{i=1}^{N} g(c_i)(x_i - x_{i-1})$$

But as $|P| \to 0$, LHS $\to \int_a^b (f+g)\,dx$ and RHS $\to \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

ie $\int_a^b (f+g)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

as required.

The other one is similar.

75 If $x_i > x_{i-1}$ and $c_i \in [a,b]$ then $f(c_i)(x_i - x_{i-1}) \leq g(c_i)(x_i - x_{i-1})$ so for any partition $P = \{a = x_0 < \dots < x_N = b\}$ and $\{c_i\}$ with $c_i \in [x_{i-1}, x_i]$

$$\sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{N} g(c_i)(x_i - x_{i-1})$$

Taking the limit as $|P| \to 0$ gives

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

74 If $P = \{a = x_0 < \dots < x_N = b\}$ is a partition of $[a,b]$ and $c_i \in [x_{i-1}, x_i]$ then we have

$$\sum_{i=1}^{N} (f(c_i) + g(c_i))(x_i - x_{i-1})$$
$$= \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) + \sum_{i=1}^{N} g(c_i)(x_i - x_{i-1})$$

But as $|P| \to 0$, $LHS \to \int_a^b (f+g)\,dx$ and $RHS \to \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

ie $\int_a^b (f+g)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

as required.

The other one is similar.

75 If $x_i > x_{i-1}$ and $c_i \in [a,b]$ then $f(c_i)(x_i - x_{i-1}) \le g(c_i)(x_i - x_{i-1})$ so for any partition $P = \{a = x_0 < \dots < x_N = b\}$ and $\{c_i\}$ with $c_i \in [x_{i-1}, x_i]$

$$\sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) \le \sum_{i=1}^{N} g(c_i)(x_i - x_{i-1})$$

Taking the limit as $|P| \to 0$ gives

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

76 As $m \leq f(x) \leq M$ and $g(x) \geq 0$ for all x, we have $m\,g(x) \leq f(x)g(x) \leq M g(x)$ so by question 75,

$$\int_a^b m\,g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \int M\,g(x)\,dx$$

and then by linearity (question 74)

$$m \int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M \int_a^b g(x)\,dx.$$

Assuming $\int_a^b g(x)\,dx \neq 0$ (otherwise the required result ~~is~~ follows trivially) then

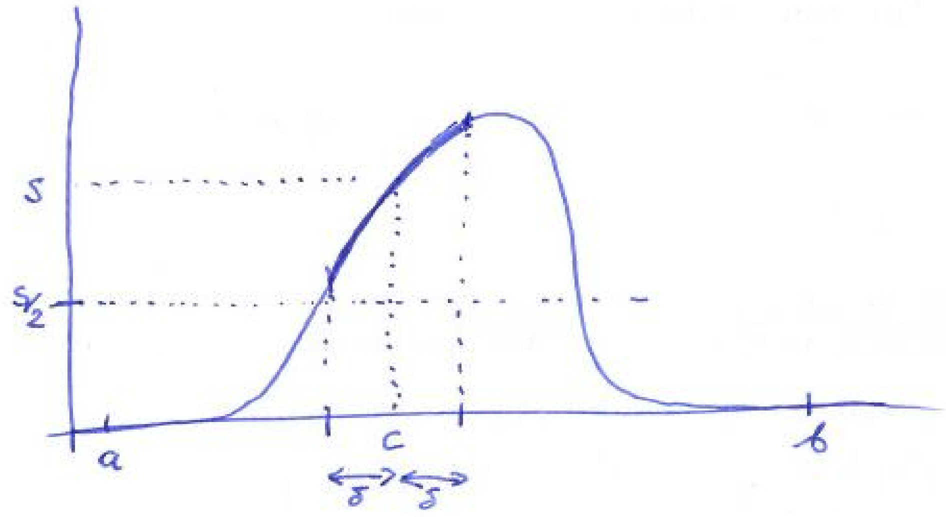
$$m \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leq M$$

Without loss of generality, we can take $m = \min_{x\in[a,b]} f(x)$, $M = \max_{x \in [a,b]} f(x)$ (as f is continuous), so that there are $\alpha, \beta \in [a,b]$ with $f(\alpha) = m$, $f(\beta) = M$. It then follows from the Intermediate Value Theorem that there is a $c \in [a,b]$ with

$$f(c) = \frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx},$$

from which the result follows.

77 f is continuous on $[a,b]$, $f(x) \geq 0$ and $f(c) > 0$ for some c

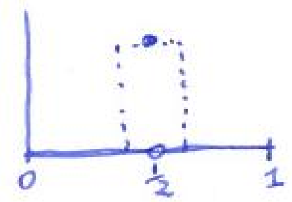


Let $f(c) = s > 0$. As f is continuous there must be a region around c on which f is greater than $s/2$, ie. there is a $\delta > 0$ such that if $x \in [c-\delta, c+\delta]$ then $f(x) > s/2$, ie $\inf\{f(x) \mid x \in [c-\delta, c+\delta]\} > s/2$. So taking the partition $P = \{a < c-\delta < c+\delta < b\}$ we have the lower sum satisfying

$$L_P(f) > 0 + s/2 \times 2\delta + 0 = s\delta > 0$$

Thus $\int_a^b f(x)\,dx \geq L_P(f) > 0$. □

For a counter-example consider $f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \end{cases}$ on the interval $[0,1]$



Take the partition $P = \{0 < \frac{1}{2} - \delta < \frac{1}{2} + \delta < 1\}$ for $\delta > 0$.
This gives the upper sum $U_P(f) = 0 + 2\delta \times 1 + 0 = 2\delta$.
Clearly any lower sum is zero so $0 = L(f) \leq U(f) \leq 2\delta$ for all $\delta > 0$.
Thus $L(f) = U(f) = 0$ and $\int_0^1 f(x)\,dx = 0$. □