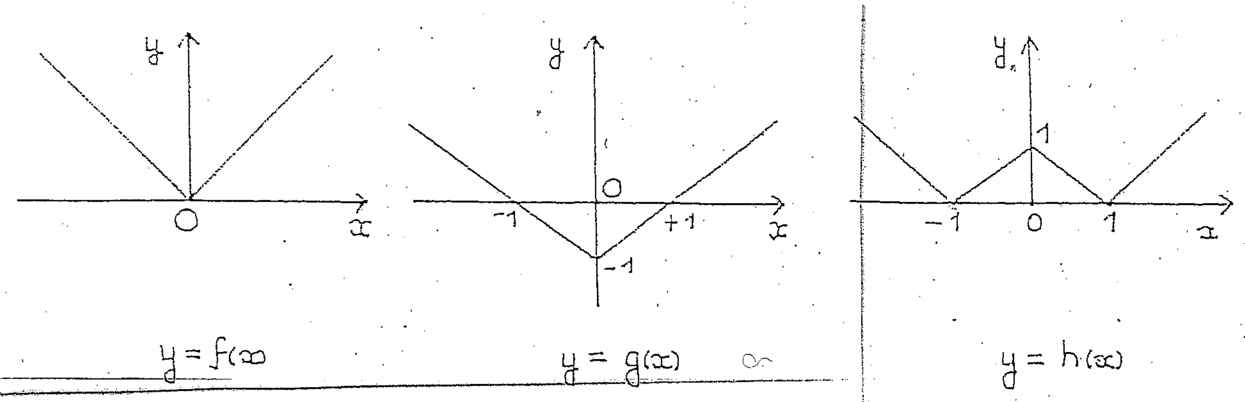
57



The function h is continuous everywhere, and differentiable everywhere except at -1, 0, 1.

58 $f(x) = \begin{cases} 0 & x<0 \\ x^7 & x \geq 0 \end{cases}$

59 (a) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

$= \lim_{h\to 0} \frac{(x^n + nx^{n-1}h + \dots + h^n) - x^n}{h} = \lim_{h\to 0} \frac{h(nx^{n-1} + \binom{n}{2}hx^{n-2} + \dots + h^{n-1})}{h}$

$= \lim_{h\to 0} \left[nx^{n-1} + h\left(\binom{n}{2}x^{n-2} + \dots + h^{n-2}\right)\right]$

$\overset{\text{a.o.l}}{=} nx^{n-1} + 0\,(\text{something}) = \underline{\underline{nx^{n-1}}}$

(b) $g(x) = x^{-n} = \frac{1}{x^n} = \frac{1}{f(x)}$, so by the quotient rule,

$g'(x) = \frac{-f'(x)}{f(x)^2} \overset{(a)}{=} \frac{-nx^{n-1}}{(x^n)^2} = \frac{-nx^{n-1}}{x^{2n}} = \underline{\underline{-nx^{-n-1}}}$

60 (a) $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{e^{a+h} - e^a}{h} = \lim_{h\to 0} \frac{e^a e^h - e^a}{h} = \lim_{h\to 0} e^a \frac{(e^h - 1)}{h}$

$\overset{\text{a.o.l + Hint}}{=} e^a.1 = \underline{\underline{e^a}}$

(b) $\sin'(a) = \lim_{h\to 0} \frac{\sin(a+h) - \sin(a)}{h} = \lim_{h\to 0} \frac{\sin(a)\cos(h) + \cos(a)\sin(h) - \sin(a)}{h}$

$= \lim_{h\to 0} \left[\sin(a) \frac{\cos(h)-1}{h} + \cos(a) \frac{\sin(h)}{h}\right] \underset{\text{a.o.l + Hint.}}{=} \sin(a) \times 0 + \cos(a) \times 1$

$= \underline{\underline{\cos(a)}}$

(c) $\cos'(a) = \lim_{h\to 0} \frac{\cos(a+h) - \cos(a)}{h} = \lim_{h\to 0} \frac{\cos(a)\cos(h) - \sin(a)\sin(h) - \cos(a)}{h}$

$= \lim_{h\to 0} \left[\cos(a) \frac{\cos(h)-1}{h} - \sin(a) \frac{\sin(h)}{h}\right] \underset{\text{a.o.l + Hint}}{=} \cos(a) \times 0 + \sin(a) \times 1$

$= \underline{-\sin(a)}$

61 $n=1$ case: $f_1'(x) = \lim_{h\to 0} \frac{f_1(x+h)-f_1(x)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1 = \underline{\underline{1x^0}}$

So true for $n=1$. Now suppose true for $n=k$, we will show it is true for $n=k+1$. Note $f_{k+1}(x) = f_k(x)\, f_1(x)$, so by the product rule:

$f_{k+1}'(x) = f_k'(x) f_1(x) + f_k(x) f_1'(x) = kx^{k-1}x + x^k.1 = kx^k + x^k$

$= \underline{(k+1)\, x^{(k+1)-1}}$ So induction step works, and true for all $n \in \mathbb{N}$.

62 (a) $k(x) := x^{1/m}$. This is defined on all of $\mathbb{R}$ if m is odd and on $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$ if m is even.

Write $y = k(x)$, so $y = x^{1/m}$ and $y^m = x$ thus $j(y) := y^m$ is the inverse to $k(x)$.

By the standard formula $k'(x) = \frac{1}{j'(y)} = \frac{1}{my^{m-1}} = \frac{1}{m(x^{1/m})^{m-1}}$

$= \frac{1}{m}\frac{1}{x^{\frac{m-1}{m}}} = \frac{1}{m}x^{\frac{1-m}{m}} = \underline{\underline{\frac{1}{m}x^{\frac{1}{m}-1}}}$

This is defined for $x>0$ if m is even and $\{x \in \mathbb{R} \mid x \neq 0\}$ if m is odd.

(b) We have $k(x) = x^{1/m}$, $f(x) = x^n$, so $l(x) := x^{n/m} = (x^n)^{1/m} = k \circ f(x)$.

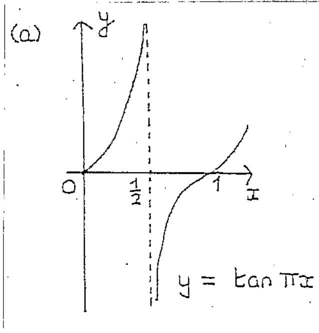
Thus by the chain rule $l'(x) = f'(x)\, k'(f(x)) = nx^{n-1} \cdot \frac{1}{m}(x^n)^{\frac{1}{m}-1}$

$= \frac{n}{m} x^{n-1} x^{\frac{n}{m}-n} = \frac{n}{m} x^{n-1+\frac{n}{m}-n} = \underline{\underline{\frac{n}{m} x^{\frac{n}{m}-1}}}$

63 $\ln(x)$ has inverse $f(x) := e^x$, so writing $y = \ln x$

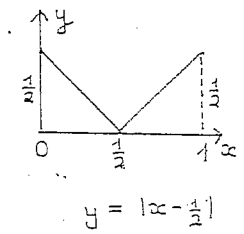
$\ln'(x) = \frac{1}{f'(y)} = \frac{1}{e^{\ln x}} = \underline{\underline{\frac{1}{x}}}$

64 (a)



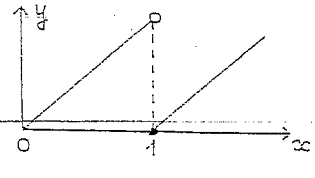
The function is neither continuous on $[0,1]$ nor differentiable on $(0,1)$, being undefined at $\frac{1}{2}$. Thus the conditions of Rolle's theorem are not satisfied.

(b)



The function is not differentiable on $(0,1)$, being undifferentiable at $\frac{1}{2}$.

(c)



The function is not continuous on $[0,1]$, being discontinuous at 1.

65. Suppose f has zeros at a and b, where $0 \leq a < b \leq 1$. Then $f(a) = f(b) = 0$, so by Rolle's theorem (conditions satisfied) $f'(c) = 3c(c-1) = 0$, for a c in $(0,1)$, which is impossible. Hence f cannot have two zeros on $[0,1]$.

66. We must find a c in $(0,1)$ such that $f'(c) = \frac{f(1)-f(0)}{1-0}$, i.e. $3c^2 + 4c + 1 = 4$, i.e. $3c^2 + 4c - 3 = 0$. Hence

$$c = (-4 \pm \sqrt{(52)})/6 = (-2 \pm \sqrt{(13)})/3.$$

Choose $c = (\sqrt{(13)} - 2)/3 \simeq 0.54$ which lies in $(0,1)$.

67. The MVT shows that, for some c in (a,b),

$$m \leq \frac{f(b)-f(a)}{b-a} = f'(c) \leq M,$$

so $m(b-a) \leq f(b) - f(a) \leq M(b-a)$ and $f(a) + m(b-a) \leq f(b) \leq f(a) + M(b-a)$

68. (a) Let $a = 0$ and $b = x$ in the MVT. Then there exists c in $(0,x)$ such that: $(f(x)-f(0))/(x-0) = f'(c)$, i.e. $\tan^{-1}x \,/\, x = 1/(1+c^2)$. Since $0 < c < x$, $1/(1+x^2) \leq 1/(1+c^2) \leq 1$. Thus

$$\frac{1}{1+x^2} < \frac{\tan^{-1}x}{x} < 1 \quad \text{and} \quad \frac{x}{1+x^2} < \tan^{-1}x < x.$$

(b) Let $a = 1$ and $b = x$ in the MVT. Then there exist c in $(1, x)$ such that: $(f(x) - f(1))/(x-1) = f'(c)$, i.e.

$$\frac{\tan^{-1}x - \pi/4}{x-1} = \frac{1}{1+c^2}.$$

Since $1 < c < x$, $1/(1+x^2) < 1/(1+c^2) < 1/(1+1) = 1/2$. Thus

$$\frac{1}{1+x^2} < \frac{\tan^{-1}x - \pi/4}{x-1} < \frac{1}{2}$$

and

$$\frac{x-1}{x^2+1} + \frac{\pi}{4} < \tan^{-1}x < \frac{1}{2}(x-1) + \frac{\pi}{4}.$$