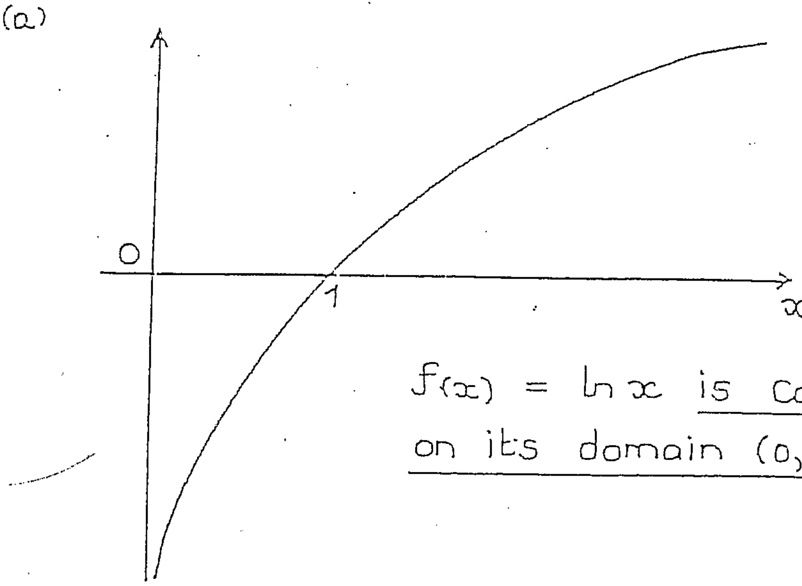
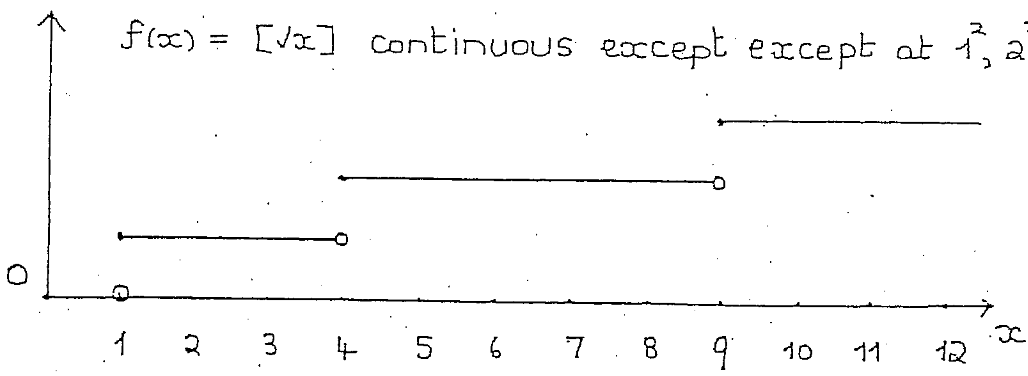
43. (a)

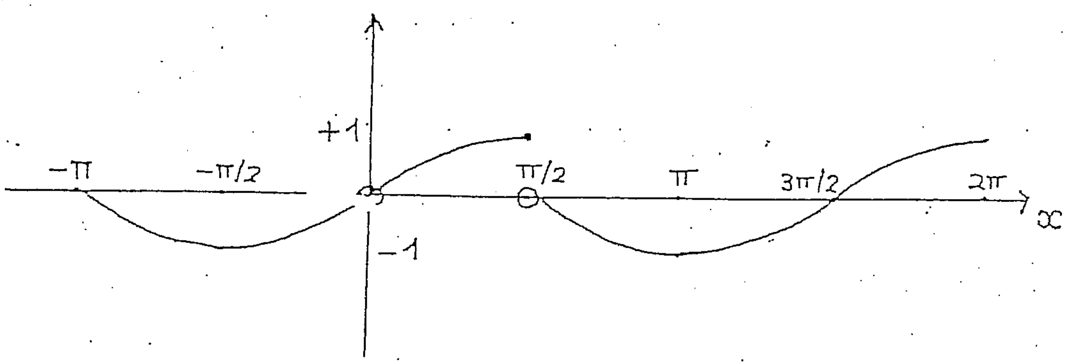


$f(x) = \ln x$ is continuous everywhere on its domain $(0,\infty)$.

(b) $f(x) = [\sqrt{x}]$ continuous except except at $1^2, 2^2, 3^2, \ldots$



(c)



$$f(x) = \begin{cases} \sin x & (x \leq \pi/2) \\ \cos x & (x > \pi/2) \end{cases}$$ is continuous everywhere except $\pi/2$.

44. Let $a \in \mathbb{R}$ and $x_n \to a$. By the algebra of limits, $1+x_n^2 \to 1+a^2$ and as $1+x_n^2, 1+a^2 \neq 0$, $1/(1+x_n^2) \to 1/(1+a^2)$, i.e. $f(x_n) \to f(a)$. Thus f is continuous at any $a \in \mathbb{R}$, and so is continuous. Now $1{\cdot}9, 1{\cdot}99, 1{\cdot}999, \ldots \to 2$ and $[(1{\cdot}9)^2], [(1{\cdot}99)^2], [(1{\cdot}999)^2], \ldots$, i.e. $3, 3, 3, \ldots$, $\to 3 \neq [2^2]$. Thus g is not continuous at 2.

45 (a) The domain of $f(x)$ is $\mathbb{R}\setminus\{0\}$. Suppose $a\in\mathbb{R}\setminus\{0\}$ and (x_n) is a sequence in $\mathbb{R}\setminus\{0\}$ with $x_n\to a$, then by the algebra of limits

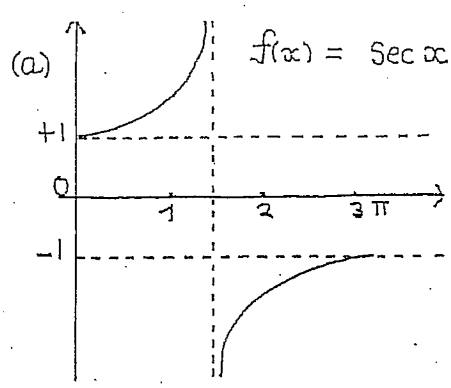
$$f(x_n)=\frac{1}{x_n}\to\frac{1}{a}=f(a)$$

as $x_n, a\neq 0$. Thus f is continuous at a.

(b) Suppose $x_n = 1+\frac{1}{n}$, then $x_n\to 1$, and $x_n\neq 1$ for any n so $g(x_n)=x_n^2 = 1+\frac{2}{n}+\frac{1}{n^2}\to 1+0+0=1$ but $g(1)=0$. Thus $x_n\to 1$ but $g(x_n)\not\to g(1)$, ie. g is <u>not</u> continuous at 1.

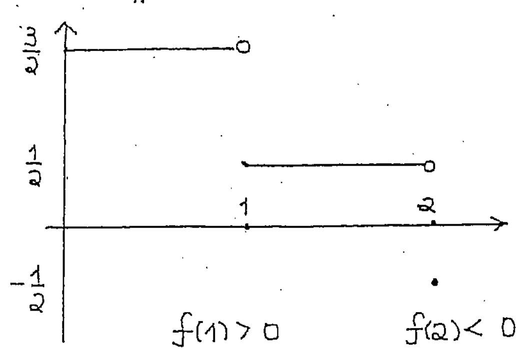
46 Let (x_n) be a sequence tending to a, then we need to show $g\circ f(x_n)\to g\circ f(a)$. However, f is continuous at a, so $f(x_n)\to f(a)$; also g is continuous at $f(a)$, and $(f(x_n))$ is a sequence tending to $f(a)$ so $g(f(x_n))\to g(f(a))$ ie. $g\circ f(x_n)\to g\circ f(a)$. □

47. (a)



Now $f(1) = \sec 1 = 1/\cos 1 > 0$ & $f(2) = \sec 2 = 1/\cos 2 < 0$. The IVT cannot be applied here as f is <u>not</u> continuous, being undefined at $\pi/2$.

(b)



We see that $f(1)=\frac{1}{2}>0$ while $f(2)=-\frac{1}{2}<0$. The IVT cannot be applied here as f is not continuous on $[1,2]$, being discontinuous at 2, for example.

48. (a) $f(2) = -1$ and $f(3) = 6$, so one answer is 2 and 3.

(b)
$f(2) < 0$, $f(3) > 0$, try $f(2.5) = 1.375 > 0$.
$f(2) < 0$, $f(2.5) > 0$, try $f(2.25) < 0$.
$f(2.25) < 0$, $f(2.5) > 0$, try $f(2.375) > 0$.
$f(2.25) < 0$, $f(2.375) > 0$, try $f(2.313) > 0$.
$f(2.25) < 0$, $f(2.313) > 0$, so to one decimal place, a root of the equation $f(x) = 0$ is $x = 2.3$.

49. (a) Let $f(x) = \cos x - 2x\sin x$. Then f is continuous on $[0, \pi/4]$. Also $f(0) = 1 > 0$ and $f(\pi/4) = (1 - \pi/2)/\sqrt{2} < 0$. By IVT, f has a root (zero) on the interval $(0, \pi/4)$.

(b) Let $f(x) = 2\tan x - 1 - \cos x$. Then f is continuous on $[0, \pi/4]$. Also $f(0) = -2 < 0$ and $f(\pi/4) = 1 - 1/\sqrt{2} > 0$. By IVT, f has a root (zero) on the interval $(0, \pi/4)$.

50. By assumption, $f(x) = \tan x - x$ is continuous on the interval $(\pi/2, 3\pi/2)$, so we may apply the intermediate value theorem and use the bisection method.

$f(3) < 0$ & $f(4.6) > 0$ so try $f(3.8) < 0$
$f(3.8) < 0$ & " so try $f(4.2)$
$f(4.2) < 0$ & " so try $f(4.4)$
$f(4.4) < 0$ & " so try $f(4.5)$
" & $f(4.5) > 0$ so try $f(4.45)$
$f(4.45) < 0$ & " so try $f(4.475)$
$f(4.475) < 0$ & " so try $f(4.4875)$
$f(4.4875) < 0$ & " so try $f(4.49375)$
" & $f(4.49375) > 0$ so try $f(4.490625)$
$f(4.490625) < 0$ & "

So $\underline{4.492}$ is within 0.02 of the solution, as the solution is between 4.490625 and 4.49375.

Note: The actual solution to 6dp is 4.493409.

PMA221 Continuity and Integration: Solutions

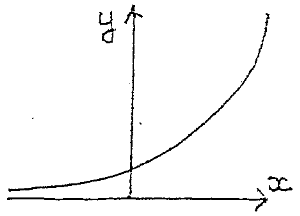
51 (a) Neither bounded above nor below, so not bounded.

(b) ~~Bounded above by 0~~, not bounded below, so not bounded.

(c) Bounded above by 1, bounded below by 0, so bounded.

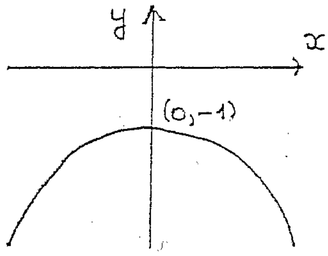
(d) Bounded above by $3+2=5$, below by $-3-2=-5$, so bounded.

52 (a)



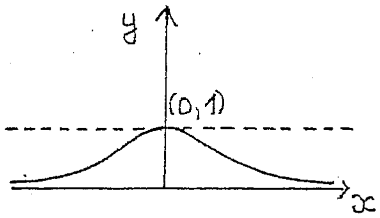
: $y = e^x$ ~ no maximum, no minimum.

(b)



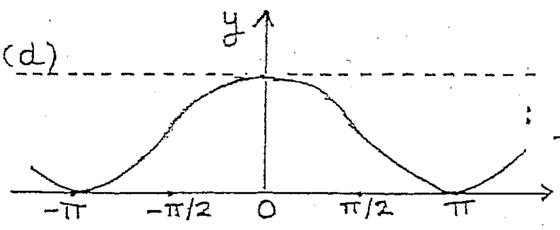
: $y = -\cosh x$ ~ maximum of -1 when $x = 0$; no minimum.

(c)



: $y = 1/(1+x^2)$ ~ maximum of 1 when $x = 0$, no minimum.

(d)



: $y = 1 + \cos x$ ~ maximum of 2, when $x = 0, \pm 2\pi, \pm 4\pi, \ldots$ minimum of 0, when $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$.

53 (a) $f(x) := \frac{1}{x}$; (b) $g(x) := \frac{1}{x+1}$; (c) $h(x) = \begin{cases} \tan(\pi x) & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2} \end{cases}$

(d) $k(x) := \begin{cases} x & x < \frac{1}{2} \\ 0 & x \geqslant \frac{1}{2} \end{cases}$; (e) The theorem requires that the interval is closed and that the function is continuous.

54 (a) $x^2/|x| = |x| \to 0$ as $x \to 0$.

(b) $x \sin(\pi/x)$ lies between $\pm x$, so limit here is 0.

(c) $\dfrac{\sin 2x}{\sin\frac{x}{2}\cos\frac{x}{2}} = \dfrac{2\sin x\cos x}{\frac{1}{2}\sin x} = 4\cos x \to -4$ as $x \to \pi$.

(d) $\dfrac{x^3-8}{x-2} = x^2 + 2x + 4 \to 2^2 + 2(2) + 4 = 12$ as $x \to 2$.