9 (a) 1, 1.414, 1.442, 1.414, 1.380, 1.348.

(b) 3, 4, 4.630, 5.063, 5.378, 5.619.

(c) 1, 1.414, 1.817, 2.213, 2.605, 2.994.

(d) 0, 0.414, 0.732, 0, 0.236, 0.449.

The matching is (a)~(C), (b)~(D), (c)~(A), (d)~(B).

10 Note $\pi = 3.1415926\ldots$. The six terms requested are: 3.1, 3.14, 3.142, 3.1416, 3.14159, 3.141593. The subsequence obtained by selecting those terms which have been rounded up, i.e. 3.142, 3.1416, 3.141593, ..., decreases to π. The remaining subsequence 3.1, 3.14, 3.14159, ..., increases to π. The first ten terms of $y_1, y_2, y_3, \ldots$ are:

3, $3 = 6/2$, $3 = 9/3$, 13/4, 16/5, 19/6, 22/7, 25/8, 28/9, 31/10.

The sequence $x_1, x_2, x_3, \ldots$ is the subsequence $y_{10}, y_{100}, y_{1000}$ of y_1, y_2, y_3.

11 The non-constant sequence 0, 1, 1, 1, ... contains neither a strictly increasing nor a strictly decreasing subsequence. Let $x_1, x_2, x_3, \ldots$ be a sequence of distinct terms. By Theorem 2.2.4, this sequence contains either an increasing subsequence or a decreasing one. The desired result follows from the observation that an increasing (decreasing) sequence of distinct terms is necessarily strictly increasing (decreasing).

12 (a) TRUE: By Theorem 2.2.4, every sequence contains either an increasing subsequence or a decreasing one. Thus if a sequence does not contain a decreasing subsequence, it must contain an increasing one.

(b) FALSE: The increasing sequence 0, 0, 0, ... contains the decreasing subsequence 0, 0, 0, ...

(c) FALSE: Suppose that $x_1, x_2, \ldots$ is an increasing sequence. Let $x_\alpha, x_\beta, \ldots$, where $\alpha, \beta, \ldots \in \mathbb{N}$ and $\alpha < \beta < \ldots$, be a subsequence of $x_1, x_2, \ldots$. Since $x_1, x_2, \ldots$ is increasing and $\alpha < \beta$, we have $x_\alpha \leq x_\beta$, which implies that the subsequence $x_\alpha, x_\beta, \ldots$ cannot be strictly decreasing.

(d) TRUE: The sequence $1, -1, 2, -2, 3, -3, \ldots$ contains both the strictly increasing subsequence $1, 2, 3, \ldots$ and the strictly decreasing subsequence $-1, -2, -3, \ldots$

13 The null sequences are (a), (b), (d)

14 Such a sequence is $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots$

15(a) The 30^{th} and 31^{st} terms are $\frac{1}{15}$ & $\frac{1}{16}$ respectively.

(b) The 22221^{th} ~~term~~ and 22222^{th} terms are both $\frac{1}{11111}$ and so sequence is decreasing, so every term after the 22222^{th} is less than $\frac{1}{11111}$.

(c) The $2M^{th}$ term is $\frac{1}{M}$ and the $(2M+1)^{th}$ term is $\frac{1}{M+1}$, as the sequence is decreasing every term after the $2M^{th}$ is less than $\frac{1}{M}$.

(d) We want a term after which all terms are less than ε. Pick an integer M so that $\frac{1}{M} \leqslant \varepsilon$, ie, any integer greater than $\frac{1}{\varepsilon}$, for instance $\frac{1}{\varepsilon}$ rounded up. Then if $n > 2M$, by (c) $x_n < \frac{1}{M}$, and so $x_n \leqq \varepsilon$. ~~So x_{2M} is such a term.~~

(e) Let $\varepsilon > 0$. Take $N > \frac{2}{\varepsilon}$ then for all $n > N$, $\alpha x_n < \frac{2}{N} < \frac{2}{2/\varepsilon} = \varepsilon$. Hence $x_n \to 0$.

16 (a) The sequence is decreasing and bigger than zero, so if $n > 100$

$$0 < x_n < x_{100} = \frac{2}{103} < \frac{2}{100} < \frac{1}{50}.$$

(b) If $n \geqslant 1000$,

$$0 < x_n < x_{1000} = \frac{2}{1003} < \frac{1}{500}.$$

(c) If $n > 20000$

$$0 < x_n < x_{20000} = \frac{2}{20003} < \frac{1}{1000}.$$

(d) Let $N > \frac{2}{\varepsilon}$ then if $n > N$

$$0 < x_n < x_N = \frac{2}{2/\varepsilon + 3} = \frac{2}{\frac{2+3\varepsilon}{\varepsilon}} = \frac{\varepsilon}{1 + \frac{3}{2}\varepsilon} < \varepsilon.$$

(e) If $\varepsilon > 0$ then take $N > \frac{2}{\varepsilon}$ then if $n > N$

$$0 < x_n < \varepsilon \quad \text{(by (d))} \quad \text{so } x_n \to 0.$$

17. Let $\varepsilon > 0$.
[Aside: We want $(\frac{1}{2})^n < \varepsilon$, since $\frac{1}{2^n} \leq \frac{1}{n}$ it suffices to have $\frac{1}{n} < \varepsilon$ ie $n > \frac{1}{\varepsilon}$.]
Choose $N > \frac{1}{\varepsilon}$. For $n > N$, we have
$$\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} \leq 1/n < \frac{1}{N} < \varepsilon.$$
Thus $(\frac{1}{2^n})$ is <u>null</u>.

18. Let $\varepsilon > 0$.
[Aside: We want $|(-1)^{n+1}/(3n^2-2)| = \frac{1}{3n^2-2} < \varepsilon$ ie $\frac{1}{\varepsilon} < 3n^2-2$ or $2 + \frac{1}{\varepsilon} < 3n^2$ so $n > \sqrt{\frac{1}{3}(2+\frac{1}{\varepsilon})}$.]
Choose $N > \sqrt{\frac{1}{3}(2+\frac{1}{\varepsilon})}$. Then for $n > N$,
$$3n^2-2 > 3N^2-2 > \frac{1}{\varepsilon} \text{ ~~and~~}$$
so
$$\left|\frac{(-1)^{n+1}}{3n^2-2}\right| = \frac{1}{3n^2-2} < \frac{1}{1/\varepsilon} = \varepsilon$$
and hence the given sequence is <u>null</u>.

19. (a) $-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots$ (b) $\pi, \pi/2, \pi/3, \ldots$ (c) $0, 0, 0, \ldots$

20. Suppose $|x_n| \leq |y_n|$ and (y_n) is null. ~~Since~~
Let $\varepsilon > 0$. Since (y_n) is null there is an N such that $|y_n| < \varepsilon$ whenever $n > N$. Thus whenever $n > N$
$$|x_n - 0| = |x_n| \leq |y_n| < \varepsilon$$
Hence (x_n) is null.

21. (a) $1, 2, 3, \ldots, n, \ldots$ (b) $-1, 1, -1, \ldots, (-1)^n, \ldots$
(c) $-\pi, \pi, -\pi, \ldots, (-1)^n\pi, \ldots$ (d) $-1, -2, -3, \ldots, -n, \ldots$

22 Let $x_n = \frac{n+3}{n}$

(a) $|x_n - 1| = \left|\frac{n+3}{n} - \frac{n}{n}\right| = \left|\frac{3}{n}\right| = \frac{3}{n}$

(b) For $n > 300$, $|x_n - 1| = \frac{3}{n} < \frac{3}{300} = \frac{1}{100}$

(c) For $n > 3{,}000$, $|x_n - 1| = 3/n < 3/3{,}000 = \frac{1}{1{,}000}$

(d) For $n > \frac{30{,}000}{7}$ $|x_n - 1| = 3/n < 3/_{30{,}000/7} = \frac{7}{10{,}000}$

(e) Let $N > 3/\varepsilon$ then for $n > N$, $\frac{1}{n} < \varepsilon/3$, so $|x_n - 1| = 3/n < 3\varepsilon/3 = \varepsilon$.

23 Let $\varepsilon > 0$. Take $N > \frac{1}{\varepsilon}$, then for all $n > N$

$$\left|\frac{\pi n - \sin(n)}{n} - \pi\right| = \frac{|\sin(n)|}{n} \leq \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

So the sequence tends to π.

24 We use the algebra of limits and the result that $1/n^\alpha \to 0$ for $\alpha > 0$.

(a) Divide top and bottom by the dominant term n^3:

$$\frac{n^3 + 1}{(2n+1)^3} = \frac{1 + (1/n^3)}{(2 + (1/n))^3} \to \frac{1+0}{(2+0)^3} = \underline{\underline{\frac{1}{8}}}$$

(b) Divide top and bottom by the dominant term n^4:

$$\frac{n^4 - 10n^3}{n^4} = 1 - \frac{10}{n} \to 1 - 0 = \underline{\underline{1}}$$

(c) $$\left(2 + 1/\sqrt{n}\right)^{10} \to (2+0)^{10} = 2^{10} = \underline{\underline{1024}}$$

25. (a) $x_n = n$, $y_n = -n$, so $x_n + y_n = 0$; (b) $x_n = y_n = (-1)^n$, so $x_n y_n = 1$;
(c) & (d) $x_n = y_n = n$, so $x_n + y_n = 2n$ and $x_n y_n = n^2$.

26. Let (x_n) converge and (y_n) diverge. Suppose $(x_n + y_n)$ converges.

By the algebra of limits $((x_n+y_n) - x_n)$ converges, being the difference of two convergent sequences, ie. (y_n) converges, which is a contradiction.

Therefore $(x_n + y_n)$ diverges.

27. Three different proofs:

(i) $1/(2n)^\alpha = 1/2^\alpha \times 1/n^\alpha \longrightarrow 1/2^\alpha \times 0 = 0$ by the algebra of limits and Thm 2.3.4: $1/n^\alpha \to 0$.

(ii) $(1/(2n)^\alpha)$ is the sequence $1/2^\alpha, 1/4^\alpha, 1/6^\alpha, \ldots$ This is a subsequence of $(1/n^\alpha)$ which converges to 0 (Thm 2.3.4), so by Thm 2.4.7 the subsequence also converges to 0.

(iii) Let $\varepsilon > 0$. Pick $N > \frac{1}{2\varepsilon^{1/\alpha}}$, then if $n > N$, $\left|1/(2n)^\alpha\right| = 1/(2n)^\alpha < \frac{1}{(2N)^\alpha} < \varepsilon$
So $1/(2n)^\alpha \to 0$.

28. Let $\varepsilon > 0$. (a) Suppose $x_n \to x$. Then there is N such that $|x_n - x| < \frac{1}{2}\varepsilon$ for $n > N$. For $n > N$,
$$|2x_n - 2x| = 2|x_n - x| < 2(\tfrac{1}{2}\varepsilon) = \varepsilon. \quad \text{Hence } 2x_n \to 2x.$$

(b) Suppose $x_n \to x$. If $a = 0$ then $ax_n = 0 \to 0 = ax$.
If $a \neq 0$ then there is N such that $|x_n - x| < \frac{1}{|a|}\varepsilon$, for $n > N$.
So for $n > N$ $|ax_n - ax| = |a||x_n - x| < |a|(\frac{1}{|a|}\varepsilon) = \varepsilon$. Hence $ax_n \to ax$.

29. We use the algebra of limits and the following limits:
$$1/n^\alpha \to 0 \ (\alpha > 0); \quad x^n \to 0 \ (-1 < x < 1); \quad x^{1/n} \to 1 \ (x > 0).$$

(a) $$\frac{2^n + 3^n}{4^n + 5^n} = \frac{(2/5)^n + (3/5)^n}{(4/5)^n + 1} \longrightarrow \frac{0+0}{0+1} = \underline{0}.$$

(b) $$\frac{2^n + 3^{2n}}{8^n + 9^n} = \frac{2^n + 9^n}{8^n + 9^n} = \frac{(2/9)^n + 1}{(8/9)^n + 1} \longrightarrow \frac{0+1}{0+1} = \underline{1}.$$

(c) $$\frac{2^{1/n} + 3^{1/n}}{4^{1/n} + 5^{1/n}} \longrightarrow \frac{1+1}{1+1} = \underline{1}.$$

(d) $$\left(3^{2+\frac{1}{n}} + 2\right)^3 = \left(9(3)^{\frac{1}{n}} + 2\right)^3 \to (9(1)+2)^3 = 11^3 = \underline{1331}.$$

(e) $$\frac{n(1/3)^{1/n} + \sqrt{n}}{2n} = \frac{1}{2}(1/3)^{1/n} + \frac{1}{2}\cdot\frac{1}{\sqrt{n}} \to \frac{1}{2}(1) + \frac{1}{2}(0) = \underline{\tfrac{1}{2}}.$$

30 To show that a sequence is decreasing, it suffices to show that the ratio of its $(n+1)$st term to its nth term is less than or equal to 1. The ratio here is:

$$\frac{((n+1)!)^2}{(2n+2)!} \div \frac{(n!)^2}{(2n)!} = \frac{((n+1)!)^2}{(n!)^2} \times \frac{(2n)!}{(2n+2)!} = \frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= \frac{n+1}{4n+2} \leq 1,$$

as desired. Thus the sequence is decreasing. It is also bounded below by 0, and so converges by Theorem 2.5.4.

31 Suppose that the positive sequence $x_1, x_2, x_3, \dots$ converges to a positive limit ℓ. Then its subsequence $x_2, x_3, x_4, \dots$ also converges to ℓ. By the algebra of limits (terms and limits are non-zero): $x_{n+1}/x_n \to \ell/\ell = 1$, as desired.

(For this second part, note a positive sequence can converge to a zero limit, e.g. $1/n \to 0$.)

(i) Take $x_n = 1/n!$ Then $x_{n+1}/x_n = n!/(n+1)! = 1/(n+1) \to 0$.

(ii) Take $x_n = 1/2^n$. Then $x_{n+1}/x_n = 1/2 \to 1/2$.

(iii) (Tricky this one!) Take (x_n) to be

$$1, 1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8}, \dots .$$

Then (x_{n+1}/x_n) is

$$1, \tfrac{1}{2}, 1, \tfrac{1}{2}, 1, \tfrac{1}{2}, 1, \tfrac{1}{2}, \dots$$

which diverges, so $\lim_{n\to\infty} x_{n+1}/x_n$ does not exist.

32(a) $a_{n+1}/a_n = \dfrac{(n+1)^p x^{(n+1)}}{n^p x^n} = \left(\dfrac{n+1}{n}\right)^p x \to 1.x = x$

(b) $0<x<1$ so there exist an N such that for all $n>N$, $0<\frac{a_{n+1}}{a_n}<1$
so for all $n>N$ $0<a_{n+1}<a_n<1$

(c) (a_n) is bounded below by 0 and eventually decreasing, so it tends to a limit $l \geq 0$

(d) Suppose $l>0$ then by Q31, $a_{n+1}/a_n \to 1$, but $a_{n+1}/a_n \to x<1$
hence $l=0$, ie $a_n \to 0$.

If $x=0$ then $a_n=0$ for all n.
If $x<0$ then $x=(-1)|x|$ but $n^p|x|^n \to 0$ so $a_n=(-1)^n n^p |x|^n \to 0$.

33 We use the algebra of limits and the result (Theorem 2.5.9) that $n^\alpha x^n \to 0$ when $\alpha \in \mathbb{R}$ and $-1<x<1$.

(a) Limit is 0. Take $\alpha = 10^{10}$ and $x=\pi/4$, noting $-1<\pi/4<1$.

(b) Divide top and bottom by the dominant term 6^n:

$$\frac{n^5 5^n + n^2 2^n}{n^7+6^n} = \frac{n^5(5/6)^n + n^2(2/6)^n}{n^7(1/6)^n+1} \to \frac{0+0}{0+1} = \underline{0}.$$

(c) Divide top and bottom by the dominant term $2^{2n}=4^n$:

$$\frac{n^{10\pi}\pi^n + n^{10e}e^n}{2^{2n}} = n^{10\pi}(\pi/4)^n + n^{10e}(e/4)^n \longrightarrow 0+0=\underline{0}.$$

34. If for every $\varepsilon>0$ we have $|x-y|<\varepsilon$ then $x=y$.
So the definition of (x_n) being ridiculously convergent to x can be rewritten as
"there exists an N such that $x_n=x$ for all $n>N$".

However $\frac{1}{n}\neq 0$ for any n, so there cannot be an N with $\frac{1}{n}=0$ for all $n>N$. Thus $(\frac{1}{n})$ is not ridiculously convergent to 0.

The sequence (x_n) converges to x if for all $\varepsilon>0$ there exists an N such that $|x_n-x|<\varepsilon$ whenever $n>N$.

The phrases "there exists an N" and "for all $\varepsilon>0$" have been switched round in the definition of ridiculously convergent.

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35 (a) True. Use Theorem 2.5.4 and the fact that any sequence of positive terms is bounded below by 0.

(b) False. The sequence $-1, -1, -1, \ldots$ converges to -1.

(c) True. An increasing sequence of negative numbers is bounded above by 0, and so converges by Completeness Axiom.

(d) False. The decreasing, convergent, positive sequence $(1/n)$ has limit 0.

36. (a) $-\frac{1}{\sqrt{n}} \leq \frac{(-1)^n}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$. Here the sandwiching sequences are null, whence given sequence has limit 0.

(b) $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$. Here again the sandwiching sequences are null, so given sequence has limit 0.

(c) $0 \leq \left(\frac{\tanh n}{n}\right)^2 \leq \frac{1}{n^2}$. Yet again the sandwiching sequences are null, and the given sequence has limit 0

(d) $-\frac{\pi}{2e^n} \leq \frac{(-1)^n \tan^{-1} n}{e^n} \leq \frac{\pi}{2e^n}$. For the last time, the sandwiching sequences are null, so the desired limit is 0.

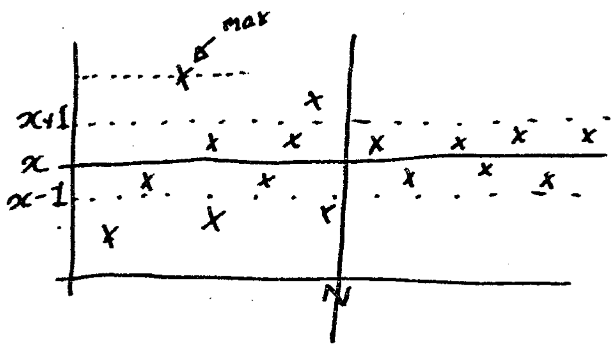
N.B. Not all sequences converge to 0!

37 (a) Set $\varepsilon = 1$ in the definition of $x_n \to x$, this then says that there is an integer N such that $|x_n - x| < 1$ whenever $n > N$.

(b) By (a), if $n > N$ then $-1 < x_n - x < 1$, ie $x - 1 < x_n < x + 1$

but if $n \leq N$ then $x_n \leq \max\{x_1, \ldots, x_N\}$

(as $x_n \in \{x_1, \ldots, x_N\}$). Thus $x_n \leq \max\{x_1, \ldots, x_N, x+1\}$.



(c) Is similar to (b)

(d) By (b) x_n is bdd above, by (c) x_n is bdd below. Thus x_n is bounded.