PMA221 Continuity and Integration: Solutions

1. When $x_1 = 1$, $x_2 = \frac{1}{2} + 1 = \frac{3}{2}$, $x_3 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}$,

$x_4 = \frac{17}{24} + \frac{12}{17} = \frac{577}{408} = 1.414215686\ldots$; $\sqrt{2} = 1.414213562\ldots$.

2. Here $dy/dx = -1/x^2$, so the requested gradient at P $(x = 1)$ is $-1/1^2 = -1$. Gradient of chord PQ is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{1/(1+10^{-n}) - 1}{1 + 10^{-n} - 1} = \frac{-10^{-n}/(1+10^{-n})}{10^{-n}}$$

$$= \frac{-1}{1+10^{-n}}.$$

For large n, Q_n is close to P and the gradient $-1/(1+10^{-n})$ of PQ_n is close to the gradient of $y = 1/x$ at P. As n increases without bound, $-1/(1+10^{-n})$ becomes closer and closer to -1. This suggests that the gradient of $y = 1/x$ at P $(x = 1)$ is -1, which supports the statement at the beginning of our answer.

3. By calculus, desired area is $\int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4} = 0.25$.
Areas of rectangles are $\frac{1}{10}((0.1)^3)$, $\frac{1}{10}((0.2)^3)$, $\ldots$, $\frac{1}{10}(1)^3$, sum is

$$\frac{1}{10}\cdot\frac{1}{10^3}(1^3+2^3+\ldots+10^3) = \frac{1}{10^4}\cdot\frac{(10)^2(11)^2}{4} = \frac{121}{400} = 0.3025.$$

For hundred rectangles, area sum is

$$\frac{1}{100}\cdot\frac{1}{(100)^3}(1^3+2^3+\ldots+100^3) = \frac{1}{10^8}\cdot\frac{(100)^2(101)^2}{4} = \frac{10201}{40000} = 0.255025$$

For thousand rectangles, area sum is

$$\frac{1}{1000}\left(\frac{1}{1000}\right)^3(1^3+2^3+\ldots+1000^3) = \frac{1}{10^{12}}\cdot\frac{(1000)^2(1001)^2}{4}$$

$$\doteq \frac{1002001}{4000000} = 0.25050025$$

The sequence of areas of rectangle sums 'seems' to be decreasing to the desired area of 0.25, which is what one might expect from simple geometrical considerations. Calculus and geometry lead to the same result here.

4 (a) finite, max 8, min 2; (b) infinite, max 66, no minimum; (c) infinite, no maximum, no minimum; (d) infinite, maximum 2000, minimum -2000; (e) finite, maximum 99, minimum 3; (f) infinite, no maximum, no minimum; (g) infinite, no maximum, no minimum; (h) infinite, maximum 3, minimum 0.

5. (a) finite, maximum 97, minimum 2; (b) infinite, no maximum, minimum 0; (c) infinite, maximum $\sqrt{5}$, minimum $-\sqrt{5}$; (d) infinite, maximum 1, no minimum; (e) this is empty set: finite, no maximum, no minimum.

6 Q4: (a) bounded above (8), bounded below (2), bounded (8); (b) bounded above (66), bounded below (10), bounded (66); (c) not bounded above, not bounded below, not bounded; (d) bounded above (2000), bounded below (-2000), bounded (2000); (e) bounded above (99), bounded below (3), bounded (99); (f) bounded above (6), not bounded below, not bounded; (g) bounded above (8), bounded below (2), bounded (8); (h) bounded above (3), bounded below (0), bounded (3). Q5: (a) bounded above (97), bounded below (2), bounded (97); (b) bounded above ($\sqrt{2}$), bounded below (0), bounded ($\sqrt{2}$); (c) bounded above ($\sqrt{5}$), bounded below ($-\sqrt{5}$), bounded ($\sqrt{5}$); (d) bounded above (1), bounded below (0), bounded (1); (e) bounded above (any real), bounded below (any real), bounded (any real).

7 (a) False – $[0,1]$; (b) True – a minimum element is always infimum of set; (c) False – $(0,1)$ has infimum 0, which is not in set; (d) False – $\{0\}$ has infimum and supremum 0; (e) False, set of rationals less than $\sqrt{2}$ has rational upper bound 2, but irrational supremum $\sqrt{2}$; (f) True – empty set.

8. (a) False – $\emptyset$; (b) False – if m is ~~the~~ a lower bound for a set E, then so is $m-1$; (c) True – if it has an infimum then it is not the empty set, it is also bounded above, so by the completeness axiom has a supremum; (d) True – any one element set such as $\{1\}$; (e) True – $(0,\pi]$; (f) False – $\{0,1\}$.