Chapter 1: **Introduction**

1. The ancient Babylonians of 1800 BC sought approximations to $\sqrt{2}$ in attempting to find the diagonal of a square of side 1. Their method was, in essence, as follows. Let $x_1 > 0$ be a first approximation. If x_1 is too small, then $2/x_1$ is too large, and vice versa. Take the next approximation x_2 to $\sqrt{2}$ to be the *average* of x_1 and $2/x_1$, i.e.,

$$x_2 = \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}x_1 + \frac{1}{x_1}.$$

The process is now repeated to produce $x_3, x_4, \ldots$, etc. Find x_4 as a fraction, given that $x_1 = 1$. Write x_4 as a decimal fraction and compare it (to within the accuracy of your pocket calculator) with the *true* value of $\sqrt{2}$.

2. Use *calculus* to find the *gradient* of the curve $y = 1/x$ at the point $P = (1, 1)$. Find, as a fraction, the gradient of the chord joining P to Q_n on the curve $y = 1/x$, where $Q_n = (1 + 10^{-n}, 1/(1 + 10^{-n}))$. Explain *briefly* how, by considering large values of n, your second answer supports your first.

3. Use calculus to find the area under the curve $y = x^3$ above the x-axis from $x = 0$ to $x = 1$. By dividing the base $[0, 1]$ into *ten* equal subintervals, and constructing vertical rectangles on and above them with height equal to the cube of the right-hand end point of the interval, find the sum of the areas of these ten rectangles. Comment. (**Hint**: Draw a diagram and use that fact that the sum of the cubes of the first n natural numbers is $n^2(n + 1)^2/4$.) Repeat with a hundred and a thousand rectangles.

4. For each of the following sets of real numbers, state whether it: (i) is finite; (ii) has a maximum element; (iii) has a minimum element. Give the maximum and minimum results when they exist.
(a) $\{2, 4, 6, 8\}$; (b) $(10, 66]$; (c) $\mathbb{Q}$; (d) $[-2000, 2000]$;
(e) $\{3, 6, 9, \ldots, 99\}$; (f) $(-\infty, 6)$; (g) $(2, 8)$; (h) $\{0\} \cup (1, 2) \cup \{3\}$.

5. Repeat the previous question for the following sets:
(a) The set of primes less than 100.
(b) The set of non-negative rationals whose square is less than 2.
(c) The set of those real numbers whose square does not exceed 5.
(d) The set of the reciprocals of the natural numbers.
(e) The set of those real numbers whose square is less than or equal to -1.

6. For each of the sets E considered in the previous two questions decide whether it is: (i) bounded above; (ii) bounded below; (iii) bounded. Give examples of upper and lower bounds where appropriate. For those sets E which are bounded, give a real number M such that $|x| \leq$ M for all x in E.

7. For each of the following statements, concerning sets of real numbers, decide whether it is true or false. For the true ones, give a brief reason; for the false ones give a counterexample.

(a) A set which has both a maximum element and a minimum element is necessarily finite.

(b) If a set has a minimum element, then it has an infimum.

(c) The infimum of a set always belongs to the set.

(d) If a set has both an infimum and a supremum, then the former is less than the latter.

(e) A non-empty set of rationals that has a rational upper bound has a rational supremum.

(f) There is a set which has 3 for an upper bound and 4 for a lower bound.

8. For each of the following statements, concerning sets of real numbers, decide whether it is true or false. For the true ones, give a brief reason; for the false ones, provide a counterexample.

(a) A set which does not have an infimum must be infinite.

(b) There is a set which has precisely one lower bound.

(c) A set that has an infimum and is bounded above has a supremum.

(d) It is possible for the maximum element of a set to be a lower bound for that set.

(e) There is a set E which has both a maximum and an infimum, but no minimum, which is such that $\sup E - \inf E = \pi$.

(f) If a set has *both* a maximum element and a minimum element, then the average of these two numbers must belong to the set.

Chapter 2: **Sequences**

9. Use a calculator to help you write down, correct to *three* decimal places (rounding *up*, where necessary), the first *six* terms of the sequences whose nth terms are:

$$\text{(a) } n^{1/n}; \quad \text{(b) } (1+2/n)^n; \quad \text{(c) } (n!)^{1/n}; \quad \text{(d) } \sqrt{n} - \lfloor \sqrt{n} \rfloor.$$

In (d), $\lfloor \cdot \rfloor$ denotes the *greatest integer part* function, i.e., $\lfloor x \rfloor$ is *the greatest integer not exceeding* x. Thus $\lfloor \pi \rfloor = 3$, $\lfloor e \rfloor = 2$ and $\lfloor -\sqrt{2} \rfloor = -2$.

Analysts use special phrases to describe the behaviour of sequences. Match the following such phrases to the sequences represented by (a), (b), (c), (d) above: (A) *tends to infinity*, (B) *oscillates*, (C) *eventually decreases to* 1, (D) *increases to* e^2. It may help to consider a few further terms of some sequences, and to sketch their *graphs* . By the *graph* of a sequence (x_n) is meant the set of isolated points (n, x_n) marked on the coordinate plane.

10. For each natural number n, let x_n be the value of π correctly rounded (up or down) to n decimal places. Give the first *six* terms of the sequence $x_1, x_2, x_3, \ldots$. Assuming that in an infinite number of cases the term is obtained by rounding up, and in an infinite number of cases it is found by rounding down, indicate how that the sequence $x_1, x_2, x_3, \ldots$ can be split into *two* subsequences, one of which is *increasing* and the other *decreasing*.

For each natural number n, let y_n be the rational number with denominator n which is closest to π. Write down the first *ten* terms of the sequence $y_1, y_2, y_3, \ldots$. What *simple* relation exists between the sequences $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$?

11. Give an example of a *non-constant* sequence that contains neither a *strictly* increasing subsequence nor a *strictly* decreasing subsequence. Show, however, that a sequence all of whose terms are *distinct* must necessarily contain either a *strictly* increasing subsequence or a *strictly* decreasing subsequence. **Hint**: Appeal to a result in the notes.

12. Precisely *two* of the following statements are true. Identify which *two* these are, giving *brief* reasons for your choice. Show *briefly* that the remaining *two* statements are false. **Hint**: In one explanation, you are expected to appeal to a result in the notes.

(a) If a sequence does *not* contain a decreasing subsequence, then it must contain an increasing subsequence.

(b) An increasing sequence can *never* contain a decreasing subsequence.

(c) An increasing sequence can contain a *strictly* decreasing subsequence.

(d) A sequence can contain both a *strictly* increasing subsequence and a *strictly* decreasing subsequence.

13. Which of the following sequences do *you* think are null?

(a) $0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, \ldots$.

(b) $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, -\frac{1}{36}, \ldots$.

(c) $0.001, 0.001, 0.001, 0.001, \ldots$.

(d) $0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots$.

(e) $\cos 1, \cos \frac{1}{2}, \cos \frac{1}{3}, \cos \frac{1}{4}, \cos \frac{1}{5}, \ldots$.

(f) $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots$.

14. Give a *specific* example of a *positive* sequence which does *not* tend to zero, but has the property that, given any positive number, however small, there is some term of the sequence less than this number.

15. Consider the sequence (x_n) beginning

$$1, 1, 1/2, 1/2, 1/3, 1/3, 1/4, 1/4, \ldots,$$

and continuing in the *obvious* way.

(a) Write down its 30th and 31st terms.

(b) Find a term *beyond which* all terms are less than $1/11111$.
(c) For a positive integer M, find a term *beyond which* all terms are less than $1/M$.
(d) For $\varepsilon > 0$ show how to find a term *beyond which* all terms are less than ε.
(e) Write down a proof that the sequence converges to zero.

16. Here you are asked to show, from the formal definition, that the sequence whose nth term is $2/(n+3)$ converges to 0.
(a) Show that *beyond* the 100th term all terms are within $1/50$ of zero.
(b) Find a term *beyond* which all terms are within $1/500$ of zero.
(c) Find a term *beyond* which all terms are within $1/10000$ of zero.
(d) Let $\varepsilon > 0$. Show how to choose N such that all the terms beyond the Nth are within ε of zero.
(e) Write down a proof that the sequence converges to zero.

17. Prove, from the ε-N–definition, that the sequence $1/2, 1/4, 1/8, \ldots, 1/2^n, \ldots$ is null. **Hint**: Use the fact that $2^n \geqslant n$.

18. Prove, from the ε-N–definition, that the sequence $\big((-1)^{n+1}/(3n^2-2)\big)$ is null.

19. Give one example each of: (a) an increasing null sequence; (b) a null sequence consisting of *only* irrational numbers; (c) a null sequence the absolute values of whose terms form an increasing sequence.

20. Show from the definition that, if (x_n) and (y_n) are two sequences such that (y_n) is null and $|x_n| \leq |y_n|$ for all n, then (x_n) is null.

21. Give examples of:
(a) a *divergent* sequence that is *increasing*;
(b) a *divergent* sequence $x_1, x_2, x_3, \ldots$ such that $x_1^2, x_2^2, x_3^2, \ldots$ is *convergent*;
(c) a *divergent* sequence $x_1, x_2, x_3, \ldots$ such that $|x_1|, |x_2|, |x_3|, \ldots$ *converges* to π;
(d) a *divergent* negative sequence $x_1, x_2, x_3, \ldots$ such that $1/x_1, 1/x_2, 1/x_3, \ldots$ is *null*.

22. Here we ask you to use the *formal* definition of convergence to show that the sequence $((n+3)/n)$ converges to 1. Let $\varepsilon > 0$.
(a) Write down the *positive* difference between the nth term and 1 .
(b) Show that *beyond* the 300th term all terms are within $\frac{1}{100}$ of 1.
(c) Find a term *beyond* which all terms are within $\frac{1}{1000}$ of 1.
(d) Find a term *beyond* which all terms are within $\frac{7}{10000}$ of 1.
(e) Show how to find N such that every term beyond the Nth is within ε of 1.

23. Use the formal definition of convergence to prove that the sequence $\left(\frac{\pi n+\sin(n)}{n}\right)$ tends to a limit.

24. Find the limits of the following sequences, giving brief reasons for your answers:

$$\text{(a)} \ \left(\frac{n^3+1}{(2n+1)^3}\right) \ ; \ \text{(b)} \ \left(\frac{n^4-10n^3}{n^4}\right) \ ; \ \text{(c)} \ \left(\left(2+\frac{1}{\sqrt{n}}\right)^{10}\right).$$

25. Give examples, *one* in each case, of *divergent* sequences (x_n) and (y_n) such that: (a) (x_n+y_n) converges; (b) (x_ny_n) converges; (c) (x_n+y_n) diverges; (d) (x_ny_n) diverges.

26. Show that if $x_1, x_2, x_3, \ldots$ is convergent and $y_1, y_2, y_3 \ldots$ is *divergent* then $x_1+y_1, x_2+y_2, x_3+y_3, \ldots$ is divergent.
Hint: Argue by contradiction, using the algebra of limits, together with the identity $y_n = (x_n+y_n) - x_n$.

27. Show that for any $\alpha > 0$ the sequence $(1/(2n)^\alpha)$ converges to 0. Can you think of another way to prove this? And another?

28. Give ε, N-proofs of the following results:
(a) if $x_n \to x$, then $2x_n \to 2x$;
(b) if $x_n \to x$, then $ax_n \to ax$ for any real number a.

29. Find the limits of the sequences whose nth terms are listed below. Give *brief* reasons for your answers.
(a) $\dfrac{2^n + 3^n}{4^n + 5^n}$; (b) $\dfrac{2^n + 3^{2n}}{8^n + 9^n}$; (c) $\dfrac{2^{1/n} + 3^{1/n}}{4^{1/n} + 5^{1/n}}$; (d) $(3^{(2+1/n)} + 2)^3$; (e) $\dfrac{n(\frac{1}{3})^{1/n} + \sqrt{n}}{2n}$.

30. Show that the sequence $((n!)^2/(2n)!)$ decreases and tends to a limit.
Hint: Consider the ratio of the $(n+1)$st term to the nth term.

31. Show that, if a positive sequence $x_1, x_2, x_3, \ldots$ converges to a *positive* limit, then $x_{n+1}/x_n \to 1$. **Hint**: Use the algebra of limits.

Give examples, one in each case, of a convergent positive sequence $x_1, x_2, x_3, \ldots$ for which the limit of x_{n+1}/x_n (i) is zero, (ii) is a half, (iii) does not exist.

32. The purpose of this question is to prove that $n^p x^n \to 0$ for $p \in \mathbb{R}$ and $-1 < x < 1$. Assume firstly that $0 < x < 1$, and write $a_n := n^p x^n$.
(a) Show that $a_{n+1}/a_n \to x$.
(b) Deduce that a_{n+1}/a_n is eventually less than one and so a_n is eventually decreasing. [Here 'eventually' means there is some N such that the statement is true for all $n > N$.]
(c) Deduce that the sequence (a_n) tends to a non-negative limit l.
(d) Use Part (a) with Question 31 to deduce that $l = 0$.
What about the cases $-1 < x \leqslant 0$?

33. Determine the limits, giving brief reasons for your answers, of the sequences whose nth terms are given below.

$$\text{(a) } n^{(10^{10})}(\pi/4)^n \ ; \ \text{(b) } \frac{n^5 5^n + n^2 2^n}{n^7 + 6^n} \ ; \ \text{(c) } \frac{n^{10\pi}\pi^n + n^{10e}e^n}{2^{2n}} \ .$$

34. The purpose of this question is to show that the order of the words in the definition of convergence is critical.

Define a sequence (x_n) to be *ridiculously-convergent* to x (this is just made up for this question) if there exists an N such that for every $\varepsilon > 0$ we have $|x_n - x| < \varepsilon$ whenever $n > N$.

Show that the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is *not* ridiculously-convergent to 0.

Give the (correct) definition for a sequence (x_n) to converge to x. What is the difference between this and the definition above?

35. For each of the following statements, decide whether it is *true* or *false*. Give brief reasons for your answers.

(a) A decreasing sequence of positive numbers is convergent.

(b) A decreasing sequence of negative numbers is divergent.

(c) An increasing sequence of negative numbers is convergent.

(d) A decreasing, convergent sequence of positive numbers has a positive limit.

36. Use the *sandwich rule* to find the limits of the following sequences:

$$\text{(a) } \left(\frac{(-1)^n}{\sqrt{n}}\right); \text{ (b) } \left(\frac{\cos n}{n}\right); \text{ (c) } \left(\left(\frac{\tanh n}{n}\right)^2\right); \text{ (d) } \left(\frac{(-1)^n \tan^{-1} n}{e^n}\right).$$

37. The purpose of this question is to prove that every convergent sequence is bounded.

Let (x_n) be a sequence converging to x.

(a) Show that there exists a number N such that $|x_n - x| < 1$ for all $n > N$.

(b) With this N show that for all n, $x_n \leqslant \max\{x_1, x_2, \ldots, x_{N-1}, x_N, x+1\}$.
[Hint: Try drawing a graph.]

(c) Show similarly that, for all n, $x_n \geqslant \min\{x_1, x_2, \ldots, x_{N-1}, x_N, x-1\}$.

(d) Deduce that (x_n) is bounded.

38. Which of the following two statements is true and which is false?

- For every number n, there is a number N such that $N > n$.
- There is a number N such that for every number n, $N > n$.

What is the difference between what is written in the two statements? What moral should you draw from this?

39. The following were all written down in the 2007/8 exam in answer to the question "What is the formal definition of a sequence (x_n) converging to a limit x?" Say what is wrong, if anything, with each of them. [You might find it useful to read them out loud to yourself.]

(a) For some $\varepsilon > 0$ there is an N such that $|x_n - x| < \varepsilon$ for $n > N$.

(b) Where $\varepsilon > 0$, for some natural number N where $n > N$ $|x_n - x| < \varepsilon$.

(c) For every positive number ε there is a term in the sequence after which all the following terms are within ε of x.

(d) For any $\varepsilon > 0$ there is some $n > N$ such that $|x_n - x| < \varepsilon$.

(e) For some $\varepsilon > 0$ there is a natural number $N < n$ such that $|x_n - x| < \varepsilon$ for all n past a certain point.

40. Are the following statements true or false?

- There exists a real number x which satisfies $x^2 \geqslant 0$.
- Every real number x satisfies $x^2 \geqslant 0$.

Which is easier to prove? Which is more useful? How would you prove that for all real numbers a and b that $a^2 + b^2 \geqslant 2ab$?

Chapter 3: **Continuity**

42. (a) Define and sketch the graph of a function $f\colon [0,1) \cup (1,2] \to [0,7]$ which satisfies $f(0) = 2 = f(2)$. Define and sketch the graph of a *discontinuous* function f_2 with these properties.

(b) Define and sketch the graph of a function $g\colon [0,2] \to [0,7]$ which extends your function f from above.

(c) Define and sketch the graph of a function $h\colon [0,2] \to [0,7]$ which extends your function f from above and satisfies $h(1) = 4$.

(d) Is it possible to find a *continuous* function $k\colon [0,2] \to [0,7]$ which extends your function f from above and satisfies $k(1) = 4$? If not, can you give a different function f for which this is possible?

(e) Give a function $m\colon [0,1] \to [0,1]$ which is discontinuous at every point. (This might require some thought.)

43. Sketch the graphs of the following functions f, and state at which points of their domains they are continuous.

$$\text{(a) } f(x) = \ln x; \qquad \text{(b) } f(x) = \lfloor \sqrt{x} \rfloor \text{ for } x \geq 0;$$

$$\text{(c) } f(x) = \begin{cases} \sin x & \text{for} \quad x \leq \frac{1}{2}\pi \\ \cos x & \text{for} \quad x > \frac{1}{2}\pi \,. \end{cases}$$

44. Show, from the sequential definition of continuity, that $f(x) = 1/(1+x^2)$ is continuous everywhere, and that $g(x) = \lfloor x^2 \rfloor$ is discontinuous at 2.

45. Show, from the sequential definition of continuity,

(a) that $f(x) = 1/x$ is continuous at every point of its domain (be careful with 0),

(b) that

$$g(x) = \begin{cases} x^2 & \text{for } x \neq 1 \\ 0 & \text{for } x = 1. \end{cases}$$

is discontinuous at 1.

46. Prove, from the sequential definition of continuity, that if the function f is continuous at the point a and the function g is continuous at $f(a)$, then the composition of f and g, $g \circ f$, defined by $g \circ f(x) := g(f(x))$ is continuous at a.

47. For each of the functions f below, $f(1) > 0$ and $f(2) < 0$, but there is no root of the equation $f(x) = 0$ between 1 and 2. Explain, with a sketch, in each case why the Intermediate Value Theorem is not appropriate here.

$$\text{(a) } f(x) = \sec x; \quad \text{(b) } f(x) = \frac{3}{2} - \lfloor x \rfloor.$$

48. Let f be the function defined by the equation $f(x) = x^3 - 3x^2 + 3x - 3$.

(a) Find two *consecutive* natural numbers such that f calculated at one of them is *positive*, and at the other is *negative*.

(b) Use the *bisection* method to find, correct to one decimal place, a root of the equation $f(x) = 0$.

49. Prove that each of the following equations has a root on the interval $(0, \frac{1}{4}\pi)$ — you may assume the continuity of standard trig functions:

$$\text{(a) } \cos x = 2x \sin x; \quad \text{(b) } 2\tan x = 1 + \cos x.$$

50. Use the bisection method to find a number within 0.002 of a root of the equation

$$\tan x - x = 0$$

between $\pi/2$ and $3\pi/2$. You may assume the continuity of $\tan x$.
Note: `maple` output is acceptable provided that it is clearly presented and clearly explained.

51. For each of the following functions $f\colon \mathbb{R} \to \mathbb{R}$, decide whether it is: (i) *bounded above*; (ii) *bounded below*; (iii) *bounded*.

(a) $f(x) = x^3$; (b) $f(x) = -x^4$;
(c) $f(x) = 1/(1+x^2)$; (d) $f(x) = 3\sin 2x + 2\cos 3x$.

52. Sketch graphs of the following functions $f\colon \mathbb{R} \to \mathbb{R}$, and discuss in each case whether or not the function has maxima or minima.

(a) $f(x) = e^x$; (b) $f(x) = -\cosh x$;
(c) $f(x) = 1/(1+x^2)$; (d) $f(x) = 1 + \cos x$.

53. (a) Give an example of a continuous function $f\colon (0,1) \to \mathbb{R}$ which is not bounded.
(b) Give an example of a continuous function $g\colon (0,1) \to \mathbb{R}$ which is bounded but has no maximum.
(c) Give an example of a function $h\colon [0,1] \to \mathbb{R}$ which is not bounded.
(d) Give an example of a function $k\colon [0,1] \to \mathbb{R}$ which is bounded but has no maximum.
(e) Give the statement of Theorem 3.3.4. Why is it that none of the above examples you have given contradicts this theorem?

54. Find the following limits, giving brief reasons for your answers.

$$\text{(a)}\ \lim_{x\to 0} \frac{x^2}{|x|}\ ;\ \text{(b)}\ \lim_{x\to 0}\ x\sin\frac{\pi}{x}\ ;\ \text{(c)}\ \lim_{x\to\pi} \frac{\sin 2x}{\sin\frac{x}{2}\cos\frac{x}{2}}\ ;\ \text{(d)}\ \lim_{x\to 2} \frac{x^3-8}{x-2}.$$

55. * Show that the sequential definition of a limit and the ε-δ definition of a limit are equivalent.

56. * Using the ε-δ definition of a limit and the idea that continuity of a function f at a point a means $\lim_{x\to a} f(x) = f(a)$, give an ε-δ definition of continuity. Use this definition to prove that $f(x) := x^2$ is continuous at $x = 2$.

Note: starred questions (*) are of a harder nature and are optional.

Chapter 4: **Differentiation**

57. Sketch the graphs of the functions:

$$f(x) := |x|; \qquad g(x) := |x| - 1; \qquad h(x) := ||x| - 1|.$$

State at which points h is (i) continuous; (ii) differentiable.

58. Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ which is six-times differentiable but not seven-times differentiable.

You have learnt the rule that for any real number p, if $f(x) = x^p$ then $f'(x) = px^{p-1}$. This is not easy to prove for any real number! We will do it below in stages in Questions 59 and 62. First you will prove it when p is a positive integer, then a negative integer, then the reciprocal of a postive integer and finally for p a rational number. How would you do it when p is an irrational number? What does x^p mean when p is irrational?

59. (a) Use the Binomial Theorem together with the formal definition of derivative to show that if n is a positive integer and $f(x) := x^n$ then $f'(x) = nx^{n-1}$.

(b) Suppose again that n is a positive integer. Use the first part of the question together with formula for the derivative of a quotient to prove that if $g(x) := x^{-n}$ then $g'(x) = (-n)x^{-n-1}$.

60. Use the formal definition of the derivative to find the derivatives of the functions below:
(a) $f(x) := e^x$;
(b) $g(x) := \sin(x)$;
(c) $k(x) := \cos(x)$.
[**Hint**: You may use the following limits:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1; \qquad \lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0; \qquad \lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

How might you prove these?]

61. Use the product formula and induction to prove that if n is a positive integer and $f_n(x) := x^n$ then ${f_n}'(x) = nx^{n-1}$ (ie. the same as in 59(a)).

62. (a) Suppose that m is a positive integer. Let $k(x) := x^{1/m}$. On what domain can k be defined? Use the formula for the derivative of the inverse of a function, together with the result of Question 59(a) to deduce the formula for the derivative $k'(x)$.

(b) Suppose now that n and m are integers with m positive. Use the previous part of this question together with the chain rule and Question 59 to find the formula of the derivative of the function $l(x) := x^{n/m}$.

63. Use the formula for the derivative of e^x together with the formula for the derivative of the inverse of a function to find the derivative of $\ln x$.

64. Each of the following functions f satisfies $f(0) = f(1)$, but there is *no* c between 0 and 1 for which $f'(c) = 0$. In each case, sketch a graph of the function and explain why Rolle's Theorem is *not* applicable:

$$\text{(a) } f(x) = \tan \pi x\ ; \quad \text{(b) } f(x) = |x - \tfrac{1}{2}|\ ; \quad \text{(c) } f(x) = x - \lfloor x \rfloor\ .$$

65. Use Rolle's Theorem, together with a contradiction argument, to show that, for *any* real number λ, the polynomial

$$f(x) \ = \ x^3 - \tfrac{3}{2}x^2 + \lambda$$

cannot have two distinct zeros on $[0, 1]$.

66. Determine a real number c which satisfies the condition required of it in the Mean Value Theorem applied to the polynomial $f(x) \ = \ x^3 + 2x^2 + x$ on $[0, 1]$.

67. The continuous function $f\colon [a,b] \to \mathbb{R}$ is differentiable at each point of the open interval (a,b), and m and M are real numbers such that $m \;\le\; f^{'}(c) \;\le\; M$ for all c in (a,b). Prove that

$$f(a) + m(b-a) \;\le\; f(b) \;\le\; f(a) + M(b-a)\ .$$

68. Let $f(x) = \tan^{-1} x$.

(a) Apply the Mean Value Theorem to f on the interval $[0,x]$, where $x > 0$, to deduce that

$$\frac{x}{1+x^2} \;<\; \tan^{-1} x \;<\; x\ .$$

(b) Apply the Mean Value Theorem to f on the interval $[1,x]$, where $x > 1$, to deduce that

$$\frac{x-1}{x^2+1} \;+\; \frac{1}{4}\pi \;<\; \tan^{-1} x \;<\; \frac{1}{2}(x-1) \;+\; \frac{1}{4}\pi\ .$$

Chapter 5: **Integration**

69. Let $P_N = \{a = x_0 < \cdots < x_N = b\}$ be an equally spaced partition of the interval $[a, b]$ given by $x_i := a + \frac{i(b-a)}{N}$. Define $f(x) := x$. Sketch a diagram illustrating the upper sum $U_{P_N}(f)$, find an expression for this upper sum, calculate the limit $\lim_{N\to\infty} U_{P_N}(f)$ and thus give an upper bound for the upper integral $U(f)$.

Similarly find an expression for the lower sum $L_{P_N}(f)$ and use it to find a lower bound for the lower integral $L(f)$.

Deduce that f is integrable on $[a, b]$ and state the integral.

70. (a) Use the addition formula for cosine to show that if α is not a multiple of 2π then

$$\sin(r\alpha) = \frac{\cos((r - \frac{1}{2})\alpha) - \cos((r + \frac{1}{2})\alpha)}{2\sin(\frac{1}{2}\alpha)}.$$

Deduce that

$$\sum_{r=1}^{N} \sin(r\alpha) = \frac{\cos(\frac{1}{2}\alpha) - \cos((N + \frac{1}{2})\alpha)}{2\sin(\frac{1}{2}\alpha)}.$$

(b) Suppose $0 < b < \frac{\pi}{2}$. Let $P_N = \{0 = x_0 < \cdots < x_N = b\}$ be an equally spaced partition of the interval $[0, b]$ given by $x_i := \frac{ib}{N}$. Draw a sketch to illustrate the upper sum $U_{P_N}(\sin)$. Find an expression for the upper sum $U_{P_N}(\sin)$ and calculate $\lim_{N\to\infty} U_{P_N}(\sin)$. Comment on your answer.
[**Hint**: You may assume $\lim_{h\to 0} \frac{\sin h}{h} = 1$.]

71. Suppose $0 < a < b$, let $P_N = \{a = x_0 < \cdots < x_N = b\}$ be a geometrically increasing partition of the interval $[a, b]$, so $x_i := aq^i$ where $q := \sqrt[N]{\frac{b}{a}}$. Consider the function $f(x) := x^k$ where $k > 0$. Observe that f is increasing on $[a, b]$. Draw a picture to illustrate the lower sum $L_{P_N}(f)$.

Show that $L_{P_N}(f) = a^{k+1}(q-1)\sum_{i=1}^{N}(q^{k+1})^{i-1}$ and apply the formula for a geometric progression to deduce

$$L_{P_N}(f) = \frac{q-1}{q^{k+1}-1}(b^{k+1} - a^{k+1}).$$

Without repeating all of the calculation, show that the upper sum $U_{P_N}(f)$ satisfies $U_{P_N}(f) = q^k L_{P_N}(f)$. Deduce that f is integrable and find $\int_a^b x^k dx$. You may assume that $\lim_{q\to 1} \frac{q-1}{q^{k+1}-1} = \frac{1}{k+1}$. [Can you prove this when k is a positive integer?]

72. Show that if f is an increasing (but not necessarily continuous) function on $[a, b]$ and if $P_N = \{a = x_0 < \cdots < x_N = b\}$ is an equally spaced partition of $[a, b]$ then the difference between the upper and lower sums is given by

$$U_{P_N}(f) - L_{P_N}(f) = \frac{1}{N}(b - a)(f(b) - f(a)).$$

Deduce that f is integrable on $[a, b]$. Can you interpret the above formula graphically? Draw a picture with the upper and lower sums on it — how do you interpret the difference as a rectangle with base $\frac{b-a}{N}$ and height $f(b) - f(a)$?

73. Show that if f is a function defined on $[a, b]$, $\delta > 0$ and P is a partition of $[a, b]$ with $|P| < \delta$, then if P' is a partition of $[a, b]$ obtained by adding a single point to P then the upper sums of P and P' satisfy

$$U_{P'}(f) > U_P(f) - \delta(M - m),$$

where as usual $M := \sup\{f(x) \mid x \in [a, b]\}$ and $m := \inf\{f(x) \mid x \in [a, b]\}$. Deduce that if P'' is a partition of $[a, b]$ obtained from P by adding μ points then

$$U_{P''}(f) > U_P(f) - \delta\mu(M - m).$$

Section 5.2.

74. Show that integration is a linear operation on functions: namely, show that if f and g are integrable functions on $[a,b]$ and C is a constant then

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

and

$$\int_a^b Cf(x)dx = C\int_a^b f(x)dx.$$

[**Hint**: Use the fact that the integral is the limit of Riemann sums.]

75. Use the fact that the integral is the limit of Riemann sums to show that if f and g are integrable functions on the interval $[a,b]$ and $f(x) \leqslant g(x)$ for all $x \in [a,b]$ then

$$\int_a^b f(x)dx \leqslant \int_a^b g(x)dx.$$

76. First Mean Value Theorem for integrals: Prove that if f and g are continuous functions on the interval $[a,b]$ with $m \leqslant f(x) \leqslant M$ for all x and $g(x) \geqslant 0$ for all x then

$$m\int_a^b g(x)dx \leqslant \int_a^b f(x)g(x)dx \leqslant M\int_a^b g(x)dx.$$

Deduce that there is a $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

77. Prove that if f is continuous on the interval $[a,b]$ and $f(x) \geqslant 0$ for all $x \in [a,b]$ and if $f(c) > 0$ for some $c \in [a,b]$ then

$$\int_a^b f(x)dx > 0.$$

Show by counter example that the result requires the assumption that f is continuous.

Chapter 6: **Revision**

78. Which of the following two statements is true and which is false?

- For every number n, there is a number N such that $N > n$.
- There is a number N such that for every number n, $N > n$.

What is the difference between what is written in the two statements? What moral should you draw from this?

79. (Q34!) The purpose of this question is to show that the order of the words in the definition of convergence is critical.

Define a sequence (x_n) to be *ridiculously-convergent* to x (this is just made up for this question) if there exists an N such that for every $\varepsilon > 0$ we have $|x_n - x| < \varepsilon$ whenever $n > N$.

(a) Give an example of a ridiculously convergent sequence. Give another example.

(b) Show that the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is *not* ridiculously-convergent to 0.

(c) Give the (correct) definition for a sequence (x_n) to converge to x. What is the difference between this and the definition above?

80. The following were all written down in the 2007/8 exam in answer to the question "What is the formal definition of a sequence (x_n) converging to a limit x?" Say what is wrong, if anything, with each of them. [You might find it useful to read them out loud to yourself.]

(a) For some $\varepsilon > 0$ there is an N such that $|x_n - x| < \varepsilon$ for $n > N$.

(b) Where $\varepsilon > 0$, for some natural number N where $n > N$ $|x_n - x| < \varepsilon$.

(c) For every positive number ε there is a term in the sequence after which all the following terms are within ε of x.

(d) For any $\varepsilon > 0$ there is some $n > N$ such that $|x_n - x| < \varepsilon$.

(e) For some $\varepsilon > 0$ there is a natural number $N < n$ such that $|x_n - x| < \varepsilon$ for all n past a certain point.

81. Are the following statements true or false?

- There exists a real number x which satisfies $x^2 \geqslant 0$.
- Every real number x satisfies $x^2 \geqslant 0$.

Which is easier to prove? Which is more useful? How would you prove that for all real numbers a and b that $a^2 + b^2 \geqslant 2ab$?

82. The following were given as answers to a question in the 2007/8 exam which asked for a statement of the Intermediate Value Theorem; say what is wrong in each case. [Try reading them out to yourself.]

Parts (a)–(c) start with the phrases "Suppose f is a real valued continuous function defined on the closed interval $[a, b]$,"

(a) ... let there exist s so that it lies between $f(a)$ and $f(b)$. Let there also exist c so that $a \leq c \leq b$. Therefore $f(c) = s$.

(b) ... then there is an s between $f(a)$ and $f(b)$ such that $a \leq c \leq b$, $f(c) = s$.

(c) ... and s lies between $f(a)$ and $f(b)$ where $a \leq c \leq b$ such that $f(c) = s$.

(d) Given an interval $[a, b]$ there exists an s such that $f(a) < s < f(b)$ and a $c \in [a, b]$ such that $f(c) = s$.

(e) If $f : [a, b] \longrightarrow \mathbb{R}$ where a and b are functions of f, and there exists another function c, where $a \leq c \leq b$ then $c = s$.

(f) A $f : [a, b] \longrightarrow \mathbb{R}$ where $a < c < b$ and $f(c) = s$. Then there is some s with $f(a) < s < f(b)$.