$\div \frac{n^3}{n^3}$ algebra of limits

1. (i) $$\frac{n^3+7n^2+n+2}{2n^3+5n+1} = \frac{1+7/n+\frac{1}{n^2}+2/n^3}{2+5/n^2+1/n^3} \to \frac{1+0+0+0}{2+0+0} = \frac{1}{2}$$ 1M 1A

using $1/n^\alpha \to 0$ for $\alpha>0$ 1

$\left(\left\lfloor 2+\frac{(-1)^n}{2n}\right\rfloor\right) = 1, 2, 1, 2, 1,$ 1 so is divergent 1

$$\frac{n^{100}+100^n}{n^{200}+100^n} = \frac{n^{100}\left(\frac{1}{100}\right)^n+1}{n^{200}\left(\frac{1}{100}\right)^n+1} \to \frac{0+1}{0+1} = 1$$ 1M 1A

as $n^\alpha x^n \to 0$ 1 for $|x|<1$

$\left(1-\frac{1}{2n}\right)^n = \left(1+\frac{-\frac{1}{2}}{n}\right)^n \to e^{-1/2}$ 1 as $\left(1+\frac{x}{n}\right)^n \to e^x$ 1

(ii) The sequence (x_n) tends to l if for all $\varepsilon>0$ 1 there exists N 1 such that $|x_n - l|<\varepsilon$ 1 for all $n>N$ 1 Lose a mark for wrong order

Let $\varepsilon>0$ 1. Let $N>\frac{1}{\sqrt{\varepsilon}}$ 1 then for $n>N$ 1

$|x_n - 0| = \frac{1}{n^2}$ 1 $< \frac{1}{N^2} = \varepsilon$ 1

Hence $x_n \to 0$. 1

2 MARKS FOR TRUE/FALSE 3 MARKS FOR REASON

2 (a) FALSE: $+1, -1, +1, -1, \ldots$ is bounded but divergent

(b) FALSE: $1, 4, 3, 6, 5, 8, 7, 10, 9 \to \infty$ but is not increasing
(up 3, down 1 sequence)

(c) TRUE: Bounded means bounded above and below
"Spanish Hotels" means every sequence contains either an increasing or decreasing subsequence.
WLOG assume it contains an ~~decreasing~~ increasing subsequence
Any upper bound for original sequence is an upper bound for the subsequence.
By the completeness axiom, bounded above & increasing implies convergent.
Therefore there is a convergent subsequence.

(d) TRUE: Argue by contradiction. Assume (x_n+y_n) convergent.
By the algebra of limits $(x_n+y_n)-(x_n)=y_n$ ~~fas~~ is convergent as x_n is also convergent
But y_n assumed not convergent, so we have a contradiction.
Therefore (x_n+y_n) is divergent.

(e) TRUE: Eg $2, 2, 3, 2, 3, 5, 2, 3, 5, 7, 2, 3, 5, 7, 11, \ldots$
This ~~gives~~ has a subsequence $3, 3, 3, \ldots$ etc for every prime.

~~4.~~ 3. (i) IVT: If $f: [a,b] \to \mathbb{R}$ is continuous and t lies between $f(a)$ & $f(b)$ then there is a $c \in [a,b]$ with $f(c) = t$. 3A

Let $f(x) = x^3 - 6x - 2$

$$\left.\begin{aligned}
f(-3) &= -27 + 18 - 2 = -11 \\
f(-2) &= -8 + 12 - 2 = +2 \\
f(-1) &= -1 + 6 - 2 = +3 \\
f(0) &= 0 + 0 - 2 = -2 \\
f(1) &= 1 - 6 - 2 = -7 \\
f(2) &= 8 - 12 - 2 = -6 \\
f(3) &= 27 - 18 - 2 = +7
\end{aligned}\right\}$$
1M ~~2~~A

Every polynomial is continuous so by the IVT there is at least one root in each interval $[-3,-2]$, $[-1,0]$ & $[2,3]$.

But a cubic can have at most 3 roots ~~on t~~ so has exactly three roots in $[-3,3]$.

(ii) $g(a) = f(a) - \alpha a$; $g(b) = f(b) - \alpha b$. So if $g(a) = g(b)$ then $f(a) - \alpha a = f(b) - \alpha b$, thus $\alpha = \dfrac{f(a) - f(b)}{a - b}$ 3A

MVT: Let $f: [a,b] \to \mathbb{R}$ be a continuous function which is differentiable at each point of (a,b), then there is some c in (a,b) such that
$$f'(c) = \frac{f(a) - f(b)}{a - b}$$
3A

Rolle's Thm: If $g: [a,b] \to \mathbb{R}$ is a continuous function which is differentiable on (a,b), ~~Apply~~ and satisfies $g(a) = g(b)$ then there is some $c \in (a,b)$ with $g'(c) = 0$. 3A

Apply Rolle's Thm to the above function, $g(x) = f(x) - \alpha$, which satisfies the conditions of the Theorem by construction - it is differentiable on (a,b) with $g'(x) = f'(x) - \alpha$. 2A

So there is $c \in (a,b)$ with $0 = g'(c) = f'(c) - \alpha$

ie with $f'(c) = \alpha = \dfrac{f(\alpha) - f(b)}{a - b}$.

4. (i) $f'(x) := \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ provided the limit exists

(ii) If $n \geq 1$ $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^n - x}{h} = \frac{x^n + nhx^{n-1} + \dots + h^n - x^n}{h}$ 1M 2A

$$= \frac{nhx^{n-1} + \binom{n}{2}h^2x^{n-2} + \dots + h^n}{h} = nx^{n-1} + h(\cdot\ \cdot)$$

$$\to nx^{n-1} \text{ as } h \to 0$$

If $n=0$ $\frac{f(x+h)-f(x)}{h} = \frac{0}{h} = 0 \to 0$ as $h \to 0$

So $f'(x) = \begin{cases} nx^{n-1} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0. \end{cases}$

(iii) $\frac{k(x+h)-k(x)}{h} = \frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$

$$= \sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin h}{h} \to \sin(x)\cdot 0 + \cos(x)\cdot 1$$

$$= \cos(x)$$

so $k'(x) = \cos(x)$.

(iv) $\frac{g(x+h)-g(x)}{h} = \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \frac{f(x) - f(x+h)}{hf(x+h)f(x)} = \frac{-\left(\frac{f(x+h)-f(x)}{h}\right)}{f(x+h)f(x)}$

$$\to \frac{-f'(x)}{(f(x))^2} \text{ as } h \to 0$$ because f is continuous at x. as f is differentiable

Limit exists, so $g(x)$ is differentiable

So $g'(x) = \frac{-f'(x)}{(f(x))^2}$

$\ell(x) = \frac{1}{\sin(x)}$ so $\ell'(x) = \frac{-\sin'(x)}{\sin^2(x)} = \frac{-\cos(x)}{\sin(x)^2}$ 2A

1

Q5

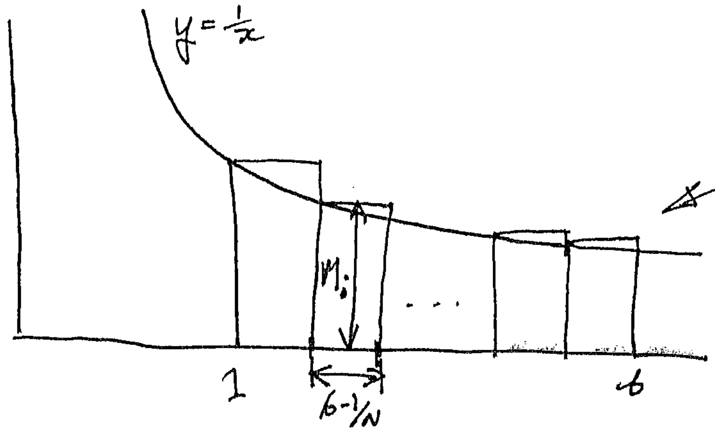
BKWK

$$U_{P_N} = \sum_{i=1}^{N} (x_i - x_{i-1}) M_i \quad \text{where } M_i = \sup\{ f(x) \mid x \in [x_{i-1}, x_i] \}$$

$$L_{P_N} = \sum_{i=1}^{N} (x_i - x_{i-1}) m_i \quad \text{where } m_i = \inf\{ f(x) \mid x \in [x_{i-1}, x_i] \}$$

3A

STD



Area of rectangles = upper sum 3A

$$U_{P_N}(f) = \sum_{i=1}^{N} \frac{b-1}{N} f(x_{i-1}) = \frac{(b-1)}{N} \sum \left(1 + \frac{(i-1)(b-1)}{N}\right)^{-1}$$

$$= \frac{(b-1)}{N} \sum_{i=1}^{N} \frac{N}{N + (i-1)(b-1)} = \underline{(b-1) \sum_{i=1}^{N} \frac{1}{N + (i-1)(b-1)}} \quad 3A$$

NSEEN

$$L_{P_N}(f) = \sum_{i=1}^{N} \frac{b-1}{N} f(x_i) = \frac{b-1}{N} \sum \left(1 + \frac{i(b-1)}{N}\right)^{-1} = \underline{(b-1) \sum_{i=1}^{N} \frac{1}{N + i(b-1)}} \quad 3A$$

$\int_1^b \frac{1}{x}\,dx = [\ln x]_1^b = \ln(b)$. 1 This is the exact area under the graph

SO $\quad L_{P_N}(f) \le \ln(b) \le U_{P_N}(f)$ 1

Taking $N = 3$, $b = 2$, $\quad U_{P_3}(f) = \sum_{i=1}^{3} \frac{1}{3 + (i-1)} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$ 2A

NSEEN

$$L_{P_3}(f) = \sum_{i=1}^{3} \frac{1}{3+i} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{37}{60} \quad 2A$$

Thus $\quad \frac{37}{60} \le \ln(b) \le \frac{47}{60}$ as required. 2A