Question 1 [2, all or none]

Bookwork

(i) Sequence (x_n) converges to x if for any $\varepsilon > 0$ $\exists N$ such that $|x_n - x| < \varepsilon \quad \forall n > N$. ①

Standard exercise, unseen.

(ii) (a) Claim $\frac{2n+1}{3n+1} \longrightarrow \frac{2}{3}$. Let $\varepsilon > 0$

Now. $\left| \frac{2n+1}{3n+1} - \frac{2}{3} \right| < \varepsilon \iff \frac{1}{3(3n+1)} < \varepsilon$ ② for calculation

$\iff \frac{1}{3}\left(\frac{1}{3\varepsilon} - 1 \right) < n$.

So take $N = \frac{1}{3}\left(\frac{1}{3\varepsilon} - 1 \right)$. Then, by above, ② for completing argument.

$\left| \frac{2n+1}{3n+1} - \frac{2}{3} \right| < \varepsilon \quad \forall n > N$.

Part bookwork; left as exercise

(b) Let $\varepsilon > 0$. Then $x + \varepsilon$ is not a lower bound of $\{x_1, x_2, x_3, \ldots\}$ ②. Hence $\exists N$ s.t. $x \leq x_N < x + \varepsilon$. If $n > N$ then, as the sequence is decreasing, $x \leq x_n \leq x_N < x + \varepsilon$

So $|x_n - x| < \varepsilon \quad \forall n > N$.

③ for an argument that works

Unseen problem.

(iii) Suppose $x_n \longrightarrow x$. Then the subsequence $x_{2n} \longrightarrow x$ ② Since $x_n > 0$ and $x > 0$, algebra of limits apply and we get $\frac{x_{2n}}{x_n} \longrightarrow \frac{x}{x} = 1$. ②

(a) $x_n = \frac{1}{2^n}$. ②

(b) $x_n = \frac{1}{n}$. ②

Question 2 (Mostly unseen, similar seen).

(i) True (1); empty set (1)

(ii) False (1); $\{1\}$ has sup $=$ inf $= 1$. (1)

(iii) False (1); $\{r \mid 0 < r < \sqrt{2},\ r \in \mathbb{Q}\}$ has sup. $\sqrt{2}$ (1)

(iv) False (1); $(0,1)$ has inf. 0 (1)

(v) False (1): $\left(\frac{\sqrt{2}}{n}\right)$ is a Cauchy sequence (2) of irrationals converging to 0.

(vi) True (1): Since $\inf E < \inf F$, $\inf F$ is no longer a lower bound of E (1) Hence $\exists\ x \in E$ s.t. $\inf E \le x < \inf F$. (1)

$x < \inf F$ implies x is a lower bd. of F (1)

(vii) True (1): Let $x = \inf E$. If $n \in \mathbb{N}$ Then $x + \frac{1}{n}$ is not a lower bd. of E. (1)

So $\exists\ x_n \in E$ such that

$$x \le x_n < x + \frac{1}{n}. \quad (1)$$

Since $\frac{1}{n} \to 0$, Sandwich rule gives

$x_n \to x$ (2)

Question 3

[2, all or none]

f is cont. at a if whenever $x_n \to a$ in the domain of f then $f(x_n) \to f(a)$.

Book work

IVT: Let $f : [a,b] \longrightarrow \mathbb{R}$ be cont. on $[a,b]$. If $s \in \mathbb{R}$ is between $f(a)$ and $f(b)$ then there is a $c \in [a,b]$ with $f(c) = s$. (2)*

* correct/true statement with weak/missing hypothesis gets 1 mark.

EVT: Let $f : [a,b] \to \mathbb{R}$ be cont. on $[a,b]$. Then f is bounded and has both a maximum and minimum on $[a,b]$ (2)*

Part bookwork; homework exercise with $[a,b]$ instead of $[0,1]$

(i) By EVT, $\exists$ $c_1, c_2 \in [0,1]$ s.t. $f(c_1) \leq f(x) \leq f(c_2)$ $\forall x \in [0,1]$. (2)

So $f([0,1]) \subseteq [f(c_1), f(c_2)]$ (1)

If $f(c_1) \leq s \leq f(c_2)$ then IVT produces a c between c_1 and c_2 such that $f(c) = s$. (3)

Therefore $f([0,1]) = [f(c_1), f(c_2)]$

Unseen; similar seen.

(ii) As in (i), let $c_1, c_2 \in [0,1]$ be such that $f(c_1) \leq f(x) \leq f(c_2)$ $\forall x \in [0,1]$.

Consider $g : [0,1] \longrightarrow \mathbb{R}$ given by $g(x) := f(x) - x$. (2)

Then g is cont. on $[0,1]$. (1)

From (i), $[0,1] \subseteq [f(c_1), f(c_2)]$

So $f(c_1) \leq 0 \leq 1 \leq f(c_2)$. (1)

Hence $g(c_1) = f(c_1) - c_1 \leq 0$ (1)

and $g(c_2) = f(c_2) - c_2 \geq 0$. (1)

By IVT, $\exists$ c between c_1 & c_2 such that $g(c) = 0$ i.e. $f(c) = c$. (2)

Such a c is of course in $[0,1]$.

Question 4

Bookwork

(i) $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists. 2

Similar seen

$f(x) = \begin{cases} x^{11} & \text{when } x \geq 0 \\ -x^{11} & \text{when } x < 0 \end{cases}$ is one example. 2

Bookwork

(ii) 2 Let $f : [a,b] \to \mathbb{R}$ be cont. on $[a,b]$, diff. on (a,b).
If $f(a) = f(b)$ then $\exists\, c \in (a,b)$ such that $f'(c) = 0$.

Similar Seen problem

(a) Suppose 1 f had zeroes at a, b where $0 \leq a < b \leq 1$.
Then $f : [a,b] \to \mathbb{R}$ satisfies the hypothesis of Rolle's Thm. 1 Thus $\exists\, c \in (a,b) \subseteq (0,1)$ s.t.
$f'(c) = 0$ 1. This gives $\underline{6c^2 - 6c = 0 \text{ i.e. } c = 0 \text{ or } 1}$ #.
3

Unseen.

Similar seen

(b) By alg. of cont & diff. f^{ns}, h is cont on $[a,b]$, diff. on (a,b) 1; it has derivative
$h'(t) = (f(b) - f(a))\, g'(t) - (g(b) - g(a))\, f'(t)$ 2

Also, $h(a) = f(b)\, g(a) - g(b)\, f(a)$ 1
$h(b) = f(b)\, g(a) - g(b)\, f(a)$ 1

Hence by Rolle's Thm, $\exists\, c \in (a,b)$ s.t. $h'(c) = 0$ 1

i.e. $(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c)$.
2 for conclusion.

Bookwork

Question 5:

$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ a partition of $[a,b]$.

Set $U(f,P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1})$ (1) (upper sum)

$L(f,P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1})$ (1) (lower sum) where

(1) for explaining M_i & m_i

$M_i := \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$; $m_i = \inf f([x_{i-1}, x_i])$

Upper integral $U(f) \overset{(1)}{=} \inf\{U(f,P) \mid P \text{ partition of } [a,b]\}$

Lower integral $L(f) \overset{(1)}{=} \sup\{L(f,P) \mid P \text{ partition of } [a,b]\}$

f is Riemann integrable if $U(f) \overset{(1)}{=} L(f)$.

Fund. inequality (2)* $L(f,P) \le L(f) \le U(f) \le U(f,P)$

* (1) for weaker/correct statements

Standard example.

(i) Let $P = \{0 = x_0 < \cdots < x_n = 1\}$ be a partition of $[0,1]$.

Then, as $f[x_{i-1}, x_i] = \{1\}$, $M_i = m_i = 1$, (1)

and $U(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1}) = 1$ (1)

$L(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1}) = 1$ (1)

So $U(f) = L(f) = 1$, (1)

Hence f is integrable (with integral 1)

Standard example.

(ii) Take $f(x) = \begin{cases} 1 & \text{when } x \in \mathbb{Q}, \ 0 \le x \le 1 \\ 0 & \text{when } x \text{ is irrational}, \ 0 \le x \le 1 \end{cases}$ (2)

Let $P = \{x_0 < \cdots < x_n\}$ be a partition of $[0,1]$.

Then, as every $[x_{i-1}, x_i]$ contains rationals and irrationals,

$$\left.\begin{aligned} M_i &= \sup f[x_{i-1}, x_i] = 1 \\ m_i &= \inf f[x_{i-1}, x_i] = 0. \end{aligned}\right\} \text{(1)}$$

$$U(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1}) = 1 \quad \text{(1)}$$

$$L(f,P) = \sum_{i=1}^{n} 0 \times (x_i - x_{i-1}) = 0 \quad \text{(1)}$$

Hence $U(f) = 1$ while $L(f) = 0$. (1)

So not integrable.

Unseen.

(iii) $f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ -1 & x \notin \mathbb{Q}. \end{cases}$ (2).

Then $f^2(x) = 1$ — so integrable; but f itself is not integrable.