The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2013–14**

Continuity and Integration **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Give the formal definition of the notion of a sequence of real numbers *converging to a limit.* ***(2 marks)***

Use the definition to prove the following two statements:

(a) The sequence $\left(\dfrac{2n+1}{3n+1}\right)$ converges. ***(5 marks)***

(b) If (x_n) is a bounded decreasing sequence with $x := \inf\{x_1, x_2, x_3, \ldots\}$, then $x_n \to x$. ***(5 marks)***

(ii) Show that if a positive sequence $x_1, x_2, x_3, \ldots$ converges to a *positive* limit, then $x_{2n}/x_n \to 1$. Theorems proved in lectures may be used without proof. ***(4 marks)***

Give examples, one in each case, of a convergent positive sequence $x_1, x_2, x_3, \ldots$ for which the limit of x_{2n}/x_n

(a) is zero,

(b) is a half.

(4 marks)

2 State which of the statements below are true and which are false. Prove those that are true, and provide counter examples for those that are false. Theorems proved in lectures may be used without proof.

(i) There is set with lower bound 10 and upper bound 5.

(ii) The supremum of a set is always strictly larger than its infimum.

(iii) A non-empty bounded set of rational numbers has a rational supremum.

(iv) The infimum of a non-empty set of positive real numbers is positive.

(v) A Cauchy sequence of irrational numbers converges to an irrational limit.

(vi) If E and F are non-empty bounded sets of real numbers such that $\inf E < \inf F$, then E contains a lower bound of F.

(vii) If E is a non-empty bounded set of real numbers then there is a sequence in E converging to the infimum of E.

(20 marks)

3 Define what it means for a real-valued function f to be *continuous* at a point a in its domain. State the *Intermediate Value Theorem* and the *Extreme Value Theorem.* ***(6 marks)***

Now let $f : [0, 1] \to \mathbb{R}$ be a continuous function.

(i) Show that $f([0, 1])$, the image of f, is also a closed and bounded interval. ***(6 marks)***

(ii) Show that if $[0, 1] \subset f([0, 1])$ then $f(c) = c$ for some $c \in [0, 1]$. ***(8 marks)***

4 (i) Describe what it means for a function f to be *differentiable* at some point a in its domain. Give an example of a function which is 10 times differentiable but not 11 times differentiable. You do not have to prove that the function you write down has the stated property. ***(4 marks)***

(ii) State Rolle's Theorem. ***(2 marks)***

(a) Show that for any real number λ, the polynomial $f(x) := 2x^3 - 3x^2 + \lambda$ cannot have two distinct zeroes on $[0, 1]$. ***(6 marks)***

(b) Let f and g be continuous on $[a, b]$ and differentiable on (a, b). By considering the function $h(t) := (f(b) - f(a))g(t) - (g(b) - g(a))f(t)$ or otherwise, show that there is a point $c \in (a, b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

(8 marks)

5 Starting with the idea of a partition of an interval, explain what it means to say that a bounded function $f : [a, b] \longrightarrow \mathbb{R}$ is *Riemann integrable* on the interval $[a, b]$. ***(6 marks)***

State inequalities relating upper sums, lower sums, upper integral and lower integral of f. ***(2 marks)***

(i) Show that the constant function $f(x) := 1$ is integrable on $[0, 1]$. ***(4 marks)***

(ii) Give, with proof, an example of a bounded function $f : [0, 1] \to \mathbb{R}$ which is not integrable. ***(6 marks)***

(iii) Give an example of a bounded function $f : [0, 1] \to \mathbb{R}$ which is not integrable but f^2 is integrable. You do not have to prove that the function you write down has the stated property. ***(2 marks)***

End of Question Paper