Instantaneous dimension of finite metric spaces via magnitude and spread

Simon Willerton
University of Sheffield

Applied Topology in Będlewo 2017

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What is the dimension?

Measure of size: magnitude [Leinster]

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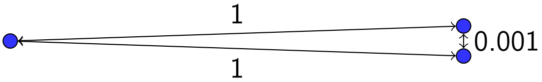

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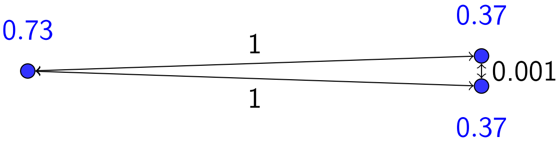

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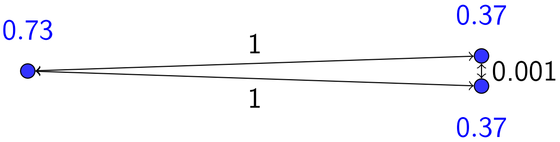
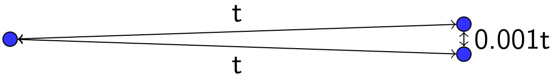


$|X| \sim 1.47$

Magnitude function: scaling a space

$tX :=$



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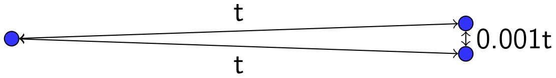
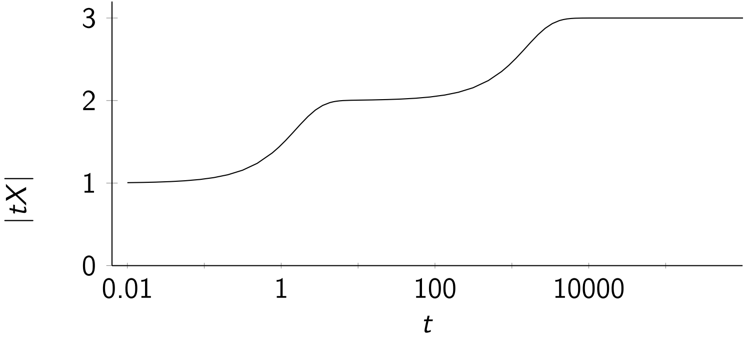
$tX :=$ 



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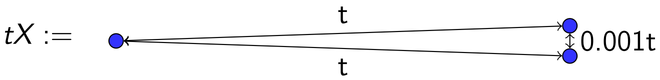

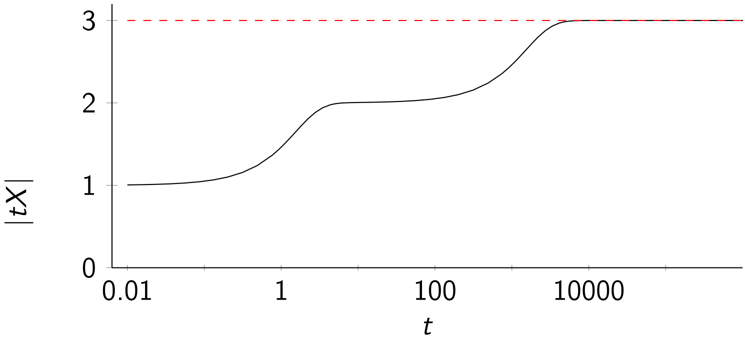

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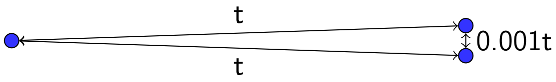
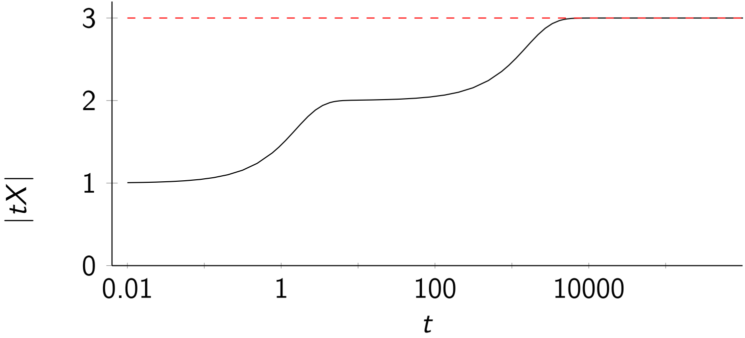
$tX :=$ 



As any space X is scaled bigger and bigger $|tX| \to N$.

Measure of size: spread [Willerton]

One parameter family of measures of size $\{\mathrm{E}_q\}_{q=0}^{\infty}$.

Closely related to magnitude but simpler and better behaved.

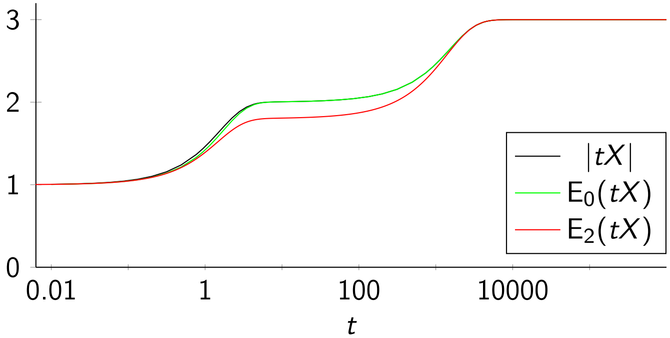
$$\mathrm{E}_q(X) := \left(\frac{1}{N} \sum_i \left(\frac{N}{\sum_j \mathrm{e}^{-\mathrm{d}(i,j)}} \right)^{1-q} \right)^{\frac{1}{1-q}}$$

In particular,

$$\mathrm{E}_0(X) = \sum_i^N \frac{1}{\sum_j \mathrm{e}^{-\mathrm{d}(i,j)}}; \qquad \mathrm{E}_2(X) = \frac{N^2}{\sum_{i,j} \mathrm{e}^{-\mathrm{d}(i,j)}}$$

If X is homogeneous then $\mathrm{E}_q(X) = |X|$ for all q.

More on spread



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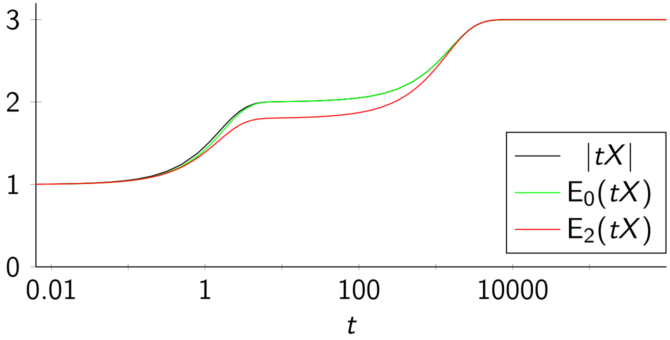

1. $1 \leq E_q(X) \leq N$;
2. $E_q(tX)$ is increasing in t;
3. $E_q(tX) \to 1$ as $t \to 0$;
4. $E_q(tX) \to N$ as $t \to \infty$.

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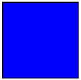

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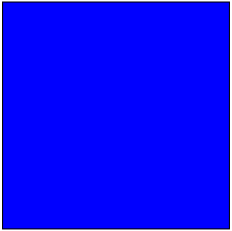

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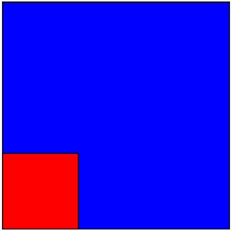

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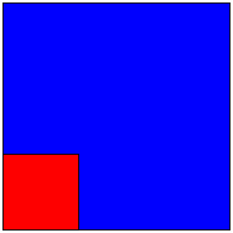

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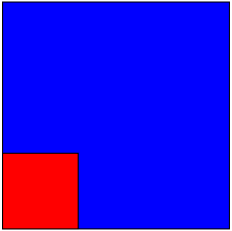

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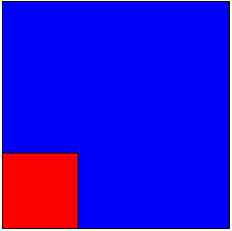

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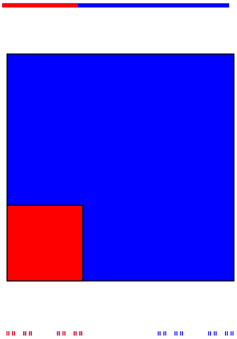

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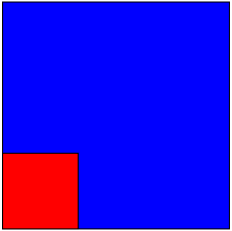

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2 times as big

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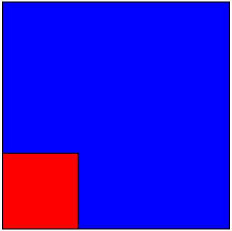
In a metric space what should happen to the size when we scale distances?

For example, triple the distances:

$3^1 = 3$ times as big



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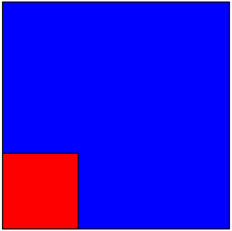

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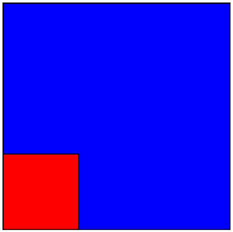

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Think of dimension as how the size changes when the distances are changed.
Given ‘size’ can see if it gives a good idea of dimension.

Instantaneous dimension

Given a notion of size S, if

$$S(tX) = a \cdot t^d$$

then we want

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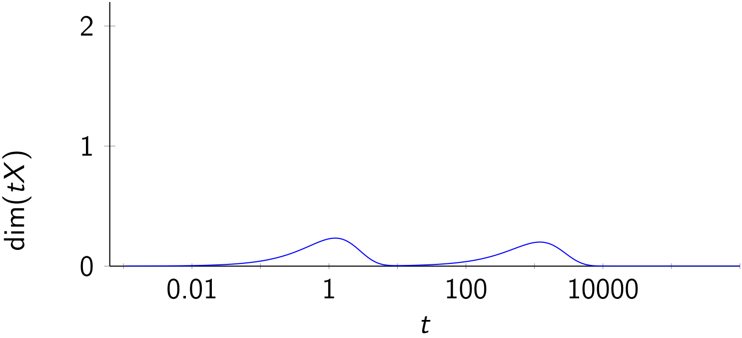
Think of $\dim_S(tX)$ as t varies as the dimension profile of X.

Can you identify these 10 point clouds from just their dimension profile????

Profile 1



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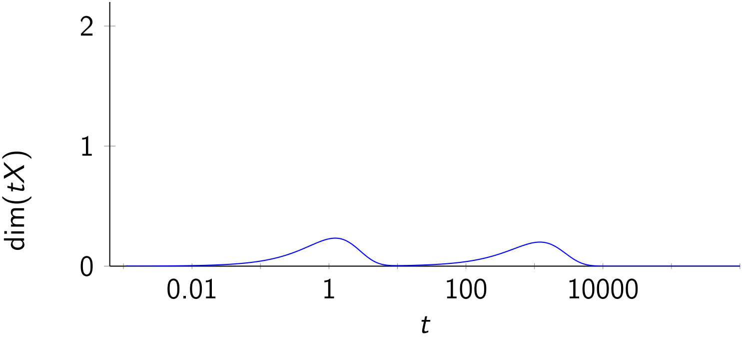
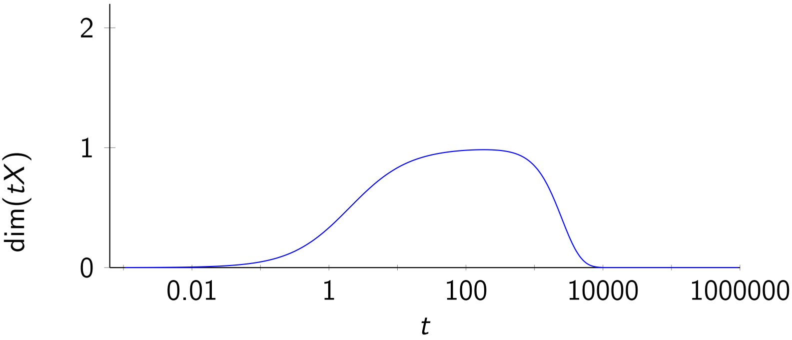


Answer: Our little 3 point space.

Profile 2



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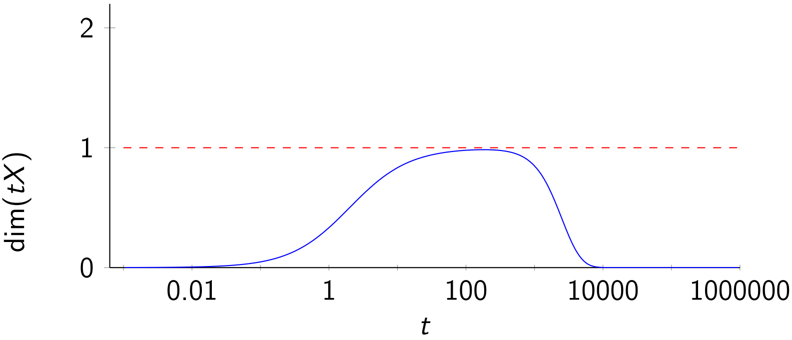

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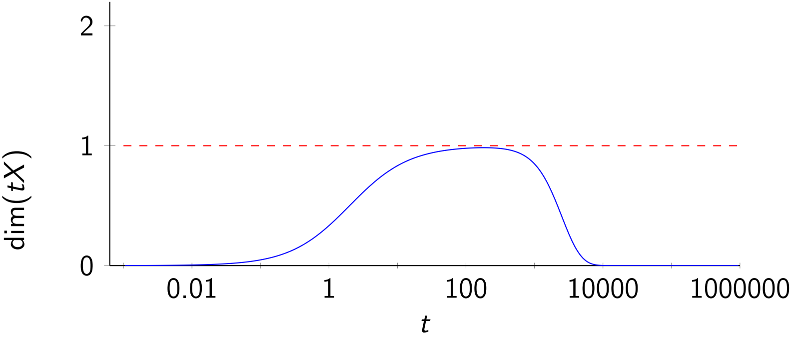
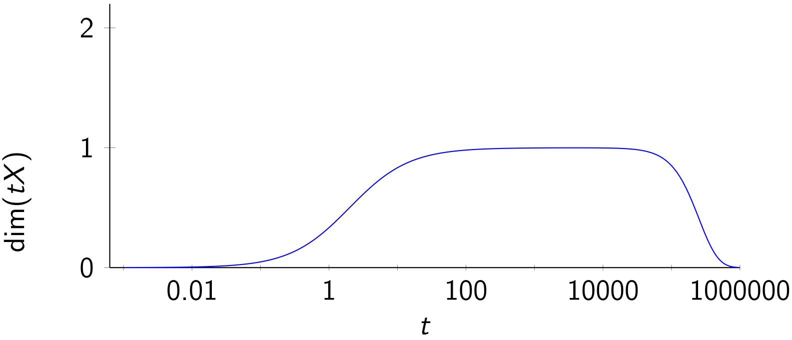


Answer: 1,000 points in the interval $[0, 1]$.

Profile 3



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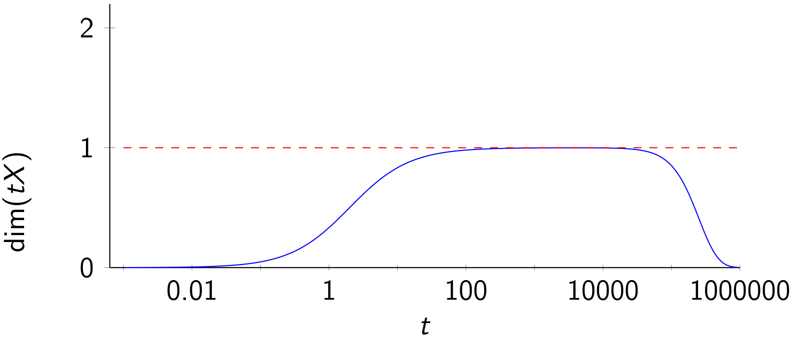

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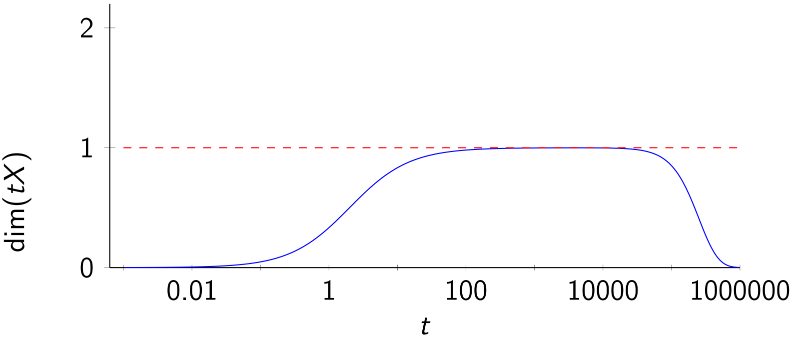
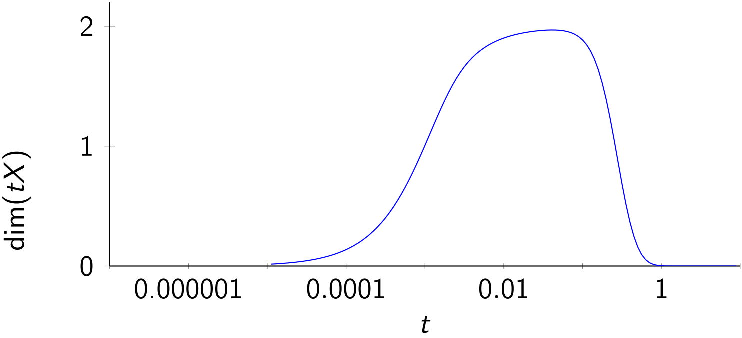
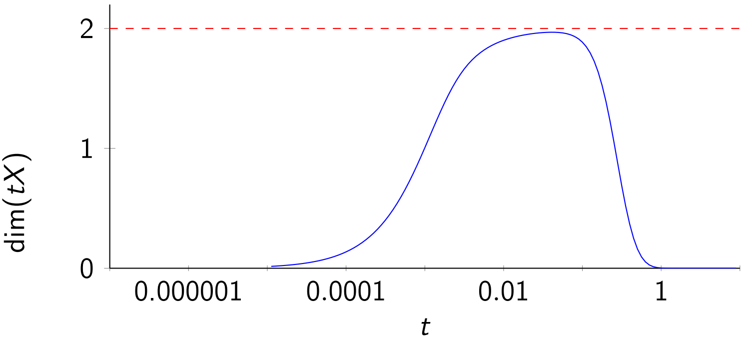


Answer: 100,000 points in the interval $[0,1]$.

Profile 4



Profile 4



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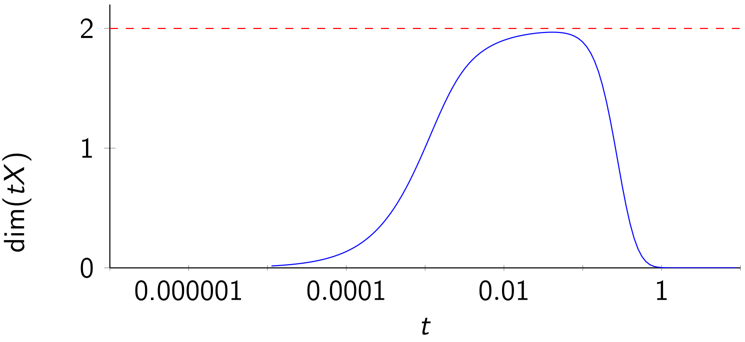
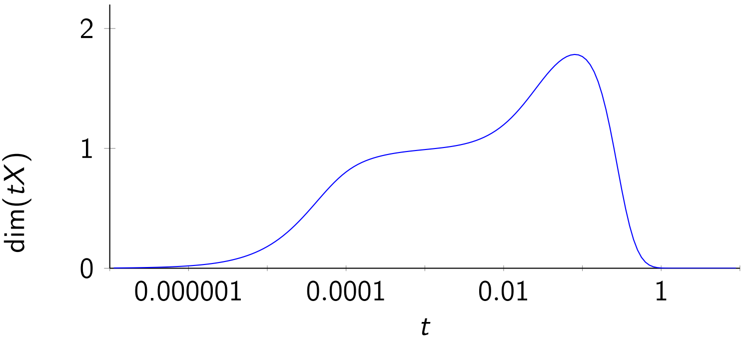


Answer: 270×270 grid of points.

Profile 5



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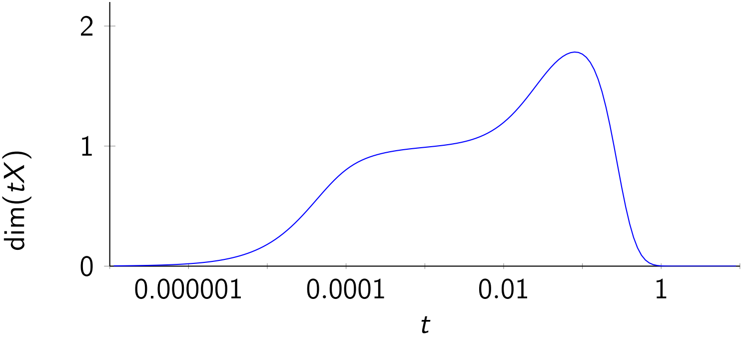
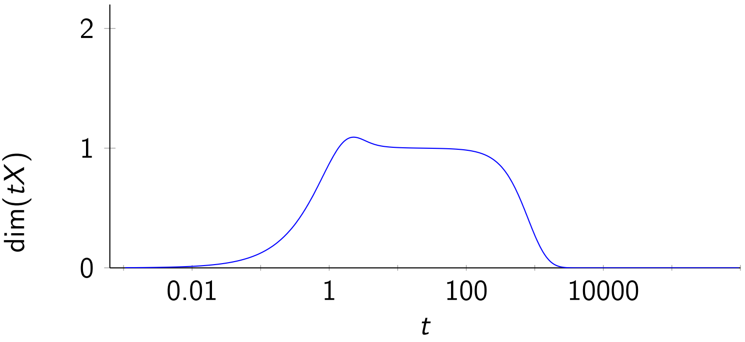
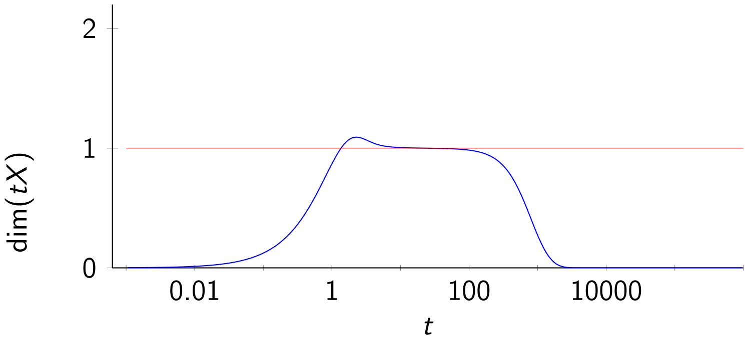


Answer: 12×6000 grid of points.

Profile 6



Profile 6



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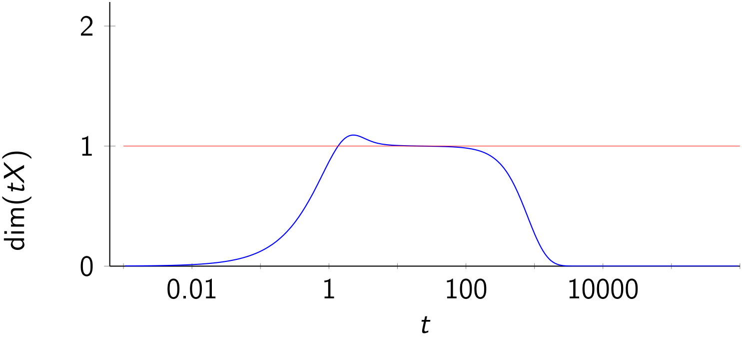
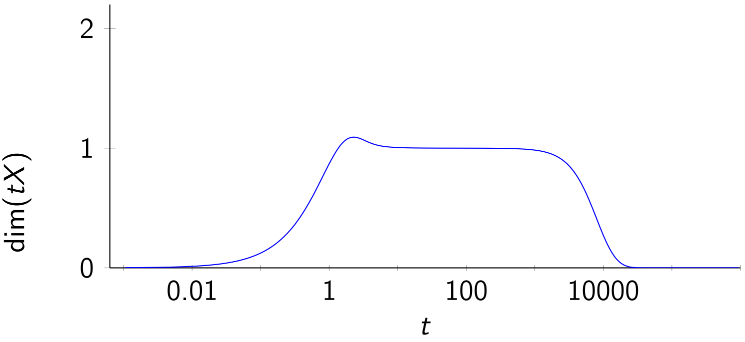


Answer: 2000 points in a circle in the plane.

Profile 7



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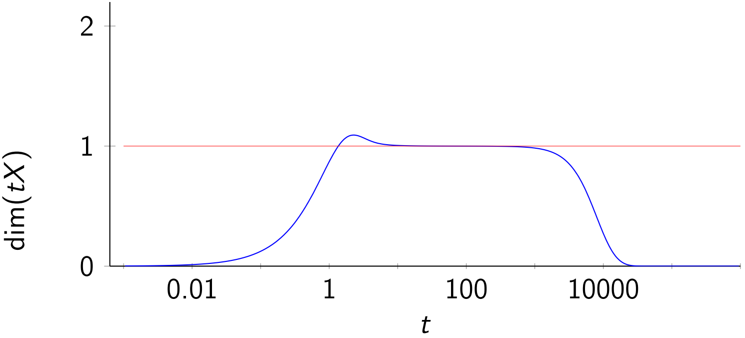

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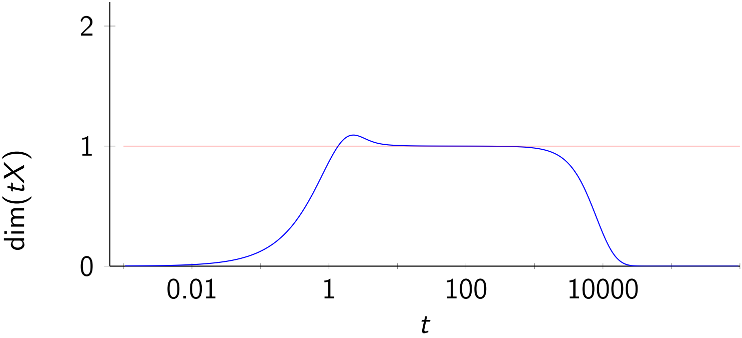
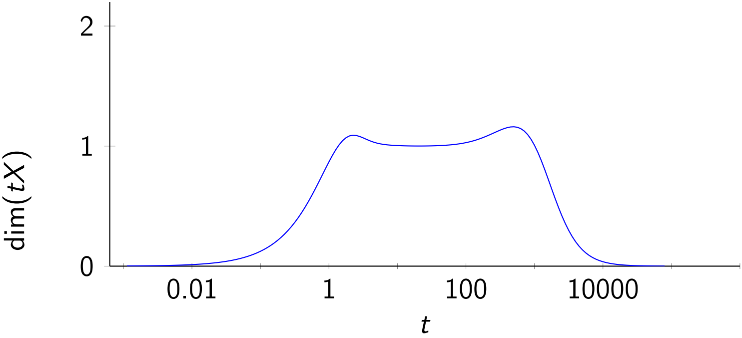


Answer: 20000 points in a circle in the plane.

Profile 8



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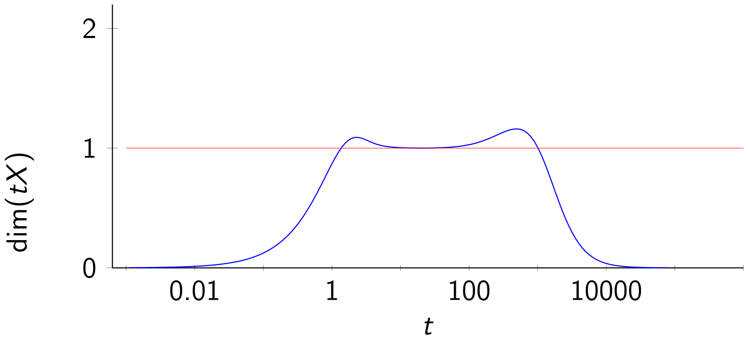

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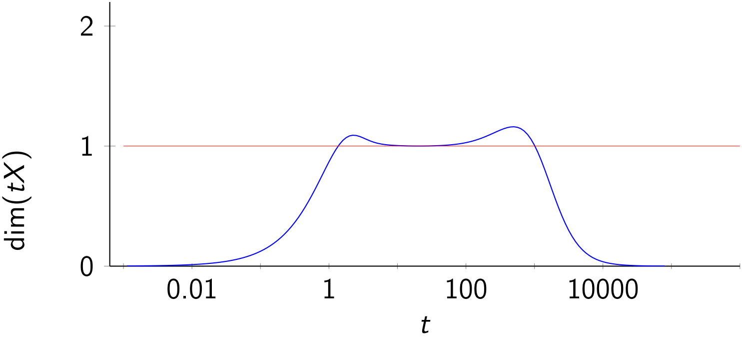
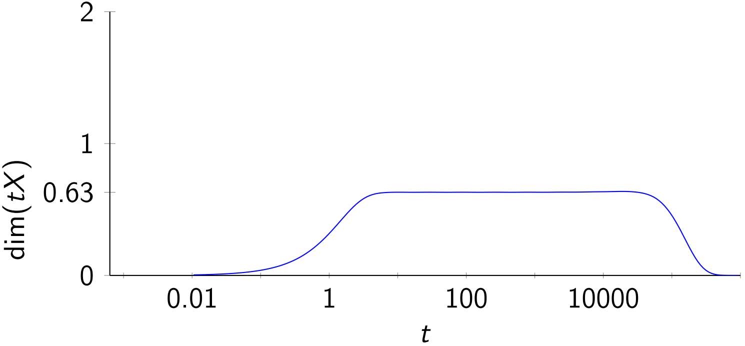


Answer: 10000 points in a ‘noisy’ circle in the plane.

Profile 9



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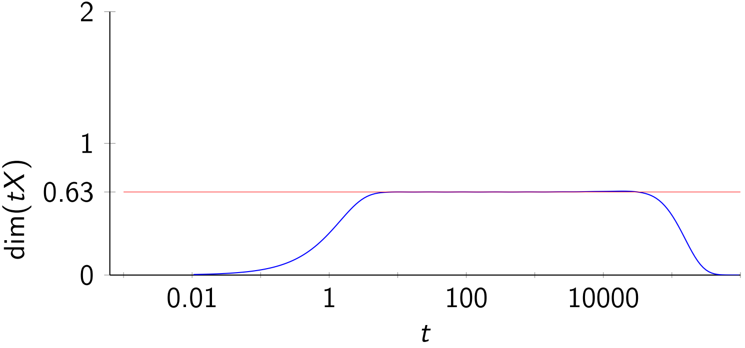

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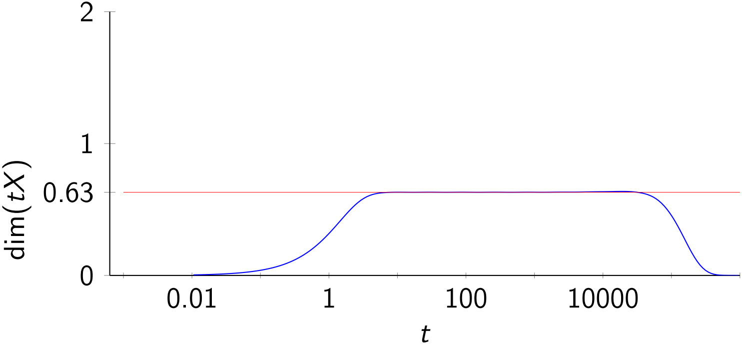

Answer: 2048 points in the Cantor set.

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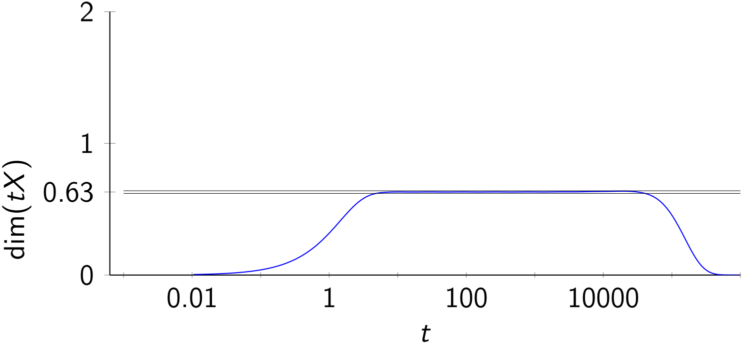

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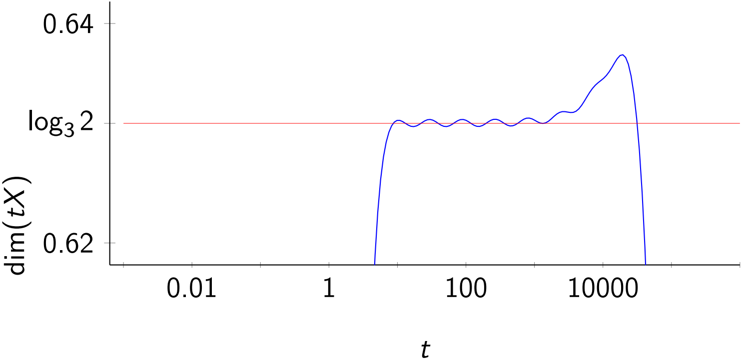

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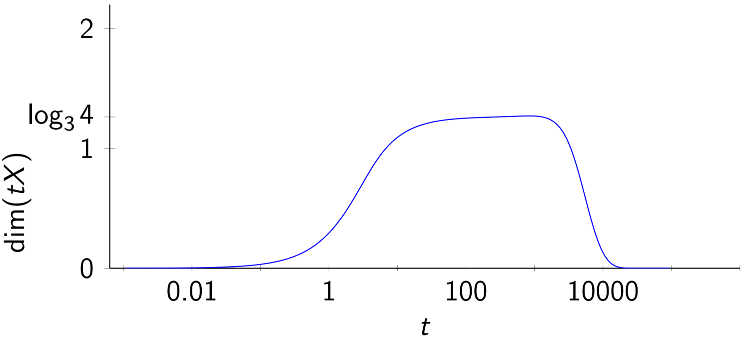

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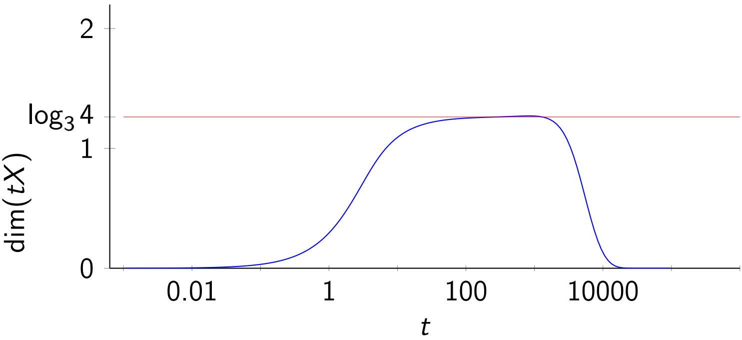

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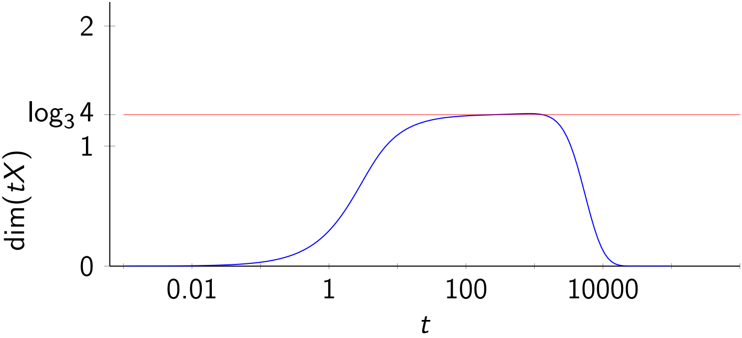

Answer: 16385 points in the Sierpinski gasket.

Challenge

Calculate the dimension profiles for some interesting data sets!