Hopf monads, Hopf algebras and diagrammatics

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QMUL
February 2020

Structure of the talk

1. Motivation

2. Monads

3. Bimonads

4. Hopf monads

5. Augmented Hopf monads

1: Motivation

Motivation stuff

X complex manifold, $D(X)$ is a symmetric monoidal category with duals.

Atiyah class gives Lie algebra $T[-1] \in D(X)$ acting on all objects in $D(X)$.

Using the diagonal $\Delta\colon X \to X \times X$ get adjoint functors

$$\Delta_*\colon D(X) \leftrightarrows D(X \times X) : \Delta^!.$$

Define $\mathcal{U} := \Delta^! \Delta_* \mathcal{O}_X$.

Then $\mathcal{U}$ is the universal enveloping algebra of $T[-1]$.

Also have $\mathcal{U} \cong \pi_* \mathcal{H}\mathrm{om}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X)$.

So $\mathcal{U}$ is an associative algebra which acts on everything in the category.

- Is $\mathcal{U}$ a Hopf algebra?
- Does $\mathcal{U} \in D(X)$ behave like $\mathbb{C}G^{\mathrm{ad}} \in \mathrm{Rep}(G)$ for G a finite group?

Reconstruction

For $\mathcal{C}$ a monoidal category with duals the end construction

$$E := \int_{V \in \mathcal{C}} V^\vee \otimes V$$

gives (if it exists) a Hopf algebra which acts on every object in the category.

E.g. if $\mathcal{C} = \mathrm{Rep}(G)$ for G a finite group then $E = \mathbb{C}G^{\mathrm{ad}}$.

However, $D(X)$ does not have enough limits and the end does not exist.

Try a different tack.

For a finite group G the diagonal $\Delta\colon G \to G \times G$ gives adjoint functors

$$\Delta_!\colon \mathrm{Rep}(G) \leftrightarrows \mathrm{Rep}(G \times G) : \Delta^*$$

with $\mathbb{C}G^{\mathrm{ad}} \cong \Delta^*\Delta_!\mathbb{C}$.

- Can we use properties of Δ^* and $\Delta_!$ to show $\Delta^*\Delta_!\mathbb{C}$ is a Hopf algebra?

2: Monads

Monads definition

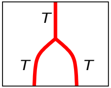
For $\mathcal{C}$ a category a monad $T\colon \mathcal{C} \to \mathcal{C}$ is an algebra in $(\mathrm{End}(\mathcal{C}), \circ, \mathrm{id})$.

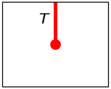
This amounts to

- endofunctor $T\colon \mathcal{C} \to \mathcal{C}$
- a product $\mu\colon T \circ T \Rightarrow T$
- unit $\iota\colon \mathrm{id} \Rightarrow T$

satisfying associativity and unit axioms.

We draw these as follows.

$\mu \equiv$ 

and $\iota \equiv$ 

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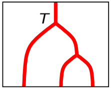
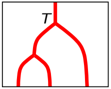
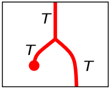
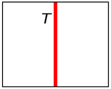
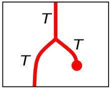
These have to satisfy the associativity and unit laws, namely



$=$ 

and 

$=$ 

$=$ 

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Example 1: monads from algebras

Suppose $(\mathcal{C}, \otimes, \mathbb{1})$ is a monoidal category and A is an algebra in $\mathcal{C}$.

Then $(A \otimes -)\colon \mathcal{C} \to \mathcal{C}$ is a monad.

E.g. For any X in $\mathcal{C}$, the product

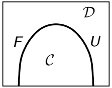
$$\mu_X \colon A \otimes A \otimes X \to A \otimes X$$

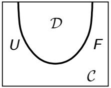
comes, in the obvious way, from the algebra product on A.

Example 2: monads from adjunctions

Suppose that $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : U$ forms an adjunction.

The counit $\epsilon\colon F \circ U \Rightarrow \mathrm{id}_{\mathcal{D}}$ and unit $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow U \circ F$ are drawn as follows.

$\epsilon \equiv$ 

; $\eta \equiv$ 

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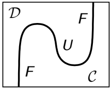
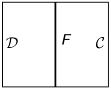
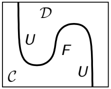
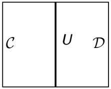
The required conditions on the unit and counit are drawn as



$=$ 

and 

$=$ 

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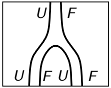
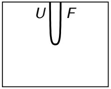
Then $U \circ F\colon \mathcal{C} \to \mathcal{C}$ forms a monad.

The multiplication and unit are formed from the unit and counit as follows:

$\mu \equiv$ 

; $\iota \equiv$ 

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Category of modules

A monad $T\colon \mathcal{C} \to \mathcal{C}$ has a category of modules $\mathcal{C}^T$.

The objects are pairs $(X \in \mathcal{C},\ \rho\colon T(X) \to X)$.

The morphisms are morphisms in $\mathcal{C}$ commuting with the actions.

E.g. For $T = (A \otimes -)$ then $\mathcal{C}^T$ is the usual category of modules of A.

3: Bimonads

Bialgebras and tensoring modules

Suppose $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ is a braided monoidal category (e.g. symmetric).

Then having a bialgebra structure on an algebra A means we can put a module structure on the tensor product of two modules:

$$A \otimes X \otimes Y \to A \otimes A \otimes X \otimes Y \to A \otimes X \otimes A \otimes Y \to X \otimes Y.$$

More precisely we can say that $\otimes$ lifts from $\mathcal{C}$ to $\mathrm{Rep}(A)$.

Lifting the tensor product for monads

Suppose we have a monad $T\colon \mathcal{C} \to \mathcal{C}$ on a monoidal category.

If we want to lift $\otimes$ from $\mathcal{C}$ to $\mathcal{C}^T$ then we cannot do it by thinking of bialgebras in $\mathrm{End}(\mathcal{C})$ as this does not have a braiding, even if $\mathcal{C}$ does.

In general, for $F_1, F_2\colon \mathcal{C} \to \mathcal{C}$

$$F_1 \circ F_2 \ncong F_2 \circ F_1.$$

Theorem (Moerdijk)

Suppose $(C, \otimes, \mathbb{1})$ *is a monoidal category and* $T\colon \mathcal{C} \to \mathcal{C}$ *is a monad. Lifts of* $\otimes$ *from* $\mathcal{C}$ *to* $\mathcal{C}^T$ *corresponds to opmonoidal structures on* T*.*

For this reason, call a monad with an opmonoidal structure a bimonad.

Opmonoidal monads, a.k.a. bimonads

Suppose $(\mathcal{C}, \otimes, \mathbb{1})$ is monoidal category.

A monad T on $\mathcal{C}$ is opmonoidal if we have (not necessarily isomorphisms):

$$T(\mathbb{1}) \to \mathbb{1} \quad \text{and} \quad T(X \otimes Y) \to T(X) \otimes T(Y) \quad \text{for } X, Y \in \mathcal{C},$$

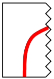
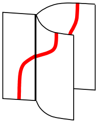
in a way compatible with the associativity and unitality of the monoidal structure and with the product and unit of the monad.

More precisely, an opmonoidal structure on T consists of natural transformations (obeying associativity and unitality conditions)

$$\sigma_2^T \colon T \circ \mathbb{1} \Rightarrow \mathbb{1} \quad \text{and} \quad \sigma_2^T \colon T \circ \otimes \Rightarrow \otimes \circ (T \times T)$$

which commute with the product μ and unit ι.

Diagrammatically:

$\sigma_0^T \equiv$ , $\qquad \sigma_2^T \equiv$ .

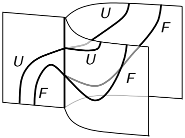
Examples of bimonads

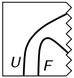
Example

Example: Suppose $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ is a braided monoidal category.
If A is a bialgebra then $(A \otimes -) \colon \mathcal{C} \to \mathcal{C}$ is a bimonad in an obvious way.

Example

Suppose $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories.
Given an adjunction $F \colon \mathcal{C} \leftrightarrows \mathcal{D} : U$ with U strong monoidal, then the monad $UF \colon \mathcal{C} \to \mathcal{C}$ is canonically a bimonad with

$\sigma_2^{UF} \equiv$ 

, and $\sigma_0^{UF} \equiv$ 

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Example

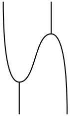
We have the adjunction $\Delta_! \colon \operatorname{Rep}(G) \leftrightarrows \operatorname{Rep}(G \times G) \colon \Delta^*$.
Pull backs are strong monoidal so $\Delta^* \Delta_!$ is a bimonad.

4: Hopf monads

Hopf algebras

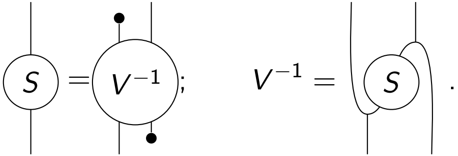
Hopf algebras are bialgebras with a certain property.
[Ignore left and right differences in this talk.]

For a bialgebra A define the fusion operator $V\colon A \otimes A \to A \otimes A$ as

$$V := (\mathrm{id} \otimes \mu) \circ (\delta \otimes \mathrm{id}) =$$  .

Theorem (Street?)

A bialgebra A is a Hopf algebra if and only if the fusion operator is invertible.

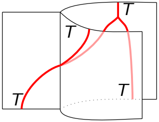


Hopf monads (Bruguières, Lack, Virelizier)

For a bimonad T on a monoidal category, the fusion operator

$$H\colon T \circ \otimes \circ (\mathrm{id} \times T) \Rightarrow \otimes \circ (T \times T)$$

is defined via



A bimonad is a Hopf monad if the fusion operator is invertible.

Theorem (BLV)

Suppose $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ is a braided monoidal category with duals. If T is a Hopf monad on $\mathcal{C}$ then $\mathcal{C}^T$ has duals.

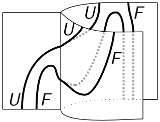
Example

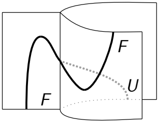
If $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ is a braided monoidal category and A is a Hopf algebra then the bimonad $(A \otimes -)$ is a Hopf monad.

Examples of Hopf monads

Example

Given an adjunction $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : U$ with U strong monoidal, then

$H :=$ 

is invertible $\iff$ $\mathbb{H} :=$ 

is invertible

In other words, if and only if we have natural isomorphisms

$$F(X \otimes U(Y)) \xrightarrow{\sim} F(X) \otimes Y \quad \text{for all } X \in \mathcal{C}, Y \in \mathcal{D}.$$

In this case we say the projection formula holds.
So UF is a Hopf monad if and only if the projection formula holds.

Example

A classic example is for $f\colon G \to K$ finite group homomorphism then

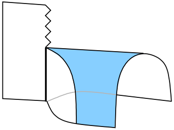
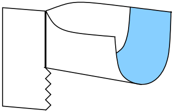
$$f_!(V \otimes f^*W) \cong f_!V \otimes W \quad \text{for all } V \in \mathrm{Rep}(G), W \in \mathrm{Rep}(K).$$

Categories with duals

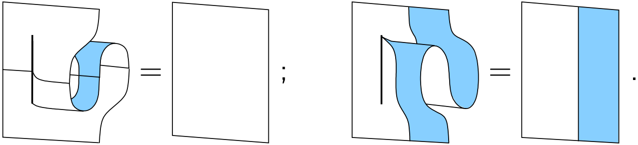
If we have a monoidal category $\mathcal{C}$ with a duals, then we have ${}^\vee\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ with coevaluation and evaluation maps.

$$\mathrm{ev}\colon X^\vee \otimes X \to \mathbb{1} \quad \text{and} \quad \mathrm{coev}\colon \mathbb{1} \to X \otimes X^\vee.$$

These give rise to so-called dinatural transformations which can be drawn as

ev $\equiv$  coev $\equiv$  .

These satisfy the so-called snake relations which become the following:

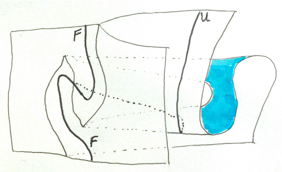


The projection formula

Theorem (Fausk, Hu, May?)

Given an adjunction $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : U$ with U strong monoidal, such that $\mathcal{C}$ and $\mathcal{D}$ have duals then the projection formula holds, so UF is a Hopf monad.

We can write down the inverse of the Hopf operator explicitly in this case.

$\mathbb{H}^{-1} =$ 

So $\Delta^*\Delta_!$ is a Hopf monad — for both $\mathrm{Rep}(G)$ and $D(X)$.

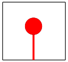
Question: Is $\Delta^*\Delta_! \cong (A \otimes -)$, as Hopf monads, for a Hopf algebra A?

In that case we would have $A = \Delta^*\Delta_!(\mathbb{1})$.

5: Augmented Hopf monads

Augmentations

An augmentation of a Hopf monad T is a bimonad map $e\colon T \to \mathrm{id}$.



An augmentation is the same as an action on each object of the category:

$$e_X\colon T(X) \to X \quad \text{for all } X.$$

And in fact gives a functor $\mathcal{C} \to \mathcal{C}^T$.

Theorem (BLV)

Let T be a Hopf monad on a braided monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$.

$$T \cong (T(\mathbb{1}) \otimes -) \text{ as Hopf monads, with } T(\mathbb{1}) \text{ a Hopf algebra}$$
$$\iff \exists e\colon T \to \mathrm{id} \text{ an augmentation compatible with the braiding } \tau.$$

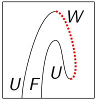
[If T is braided opmonoidal then all augmentations are compatible with τ.]

Augmentations from right inverses

Now we will use the fact that the diagonal map Δ has one sided inverses.

Theorem

If $\mathcal{C}$ and $\mathcal{D}$ are braided monoidal with duals, $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : U$ is an adjunction with U strong monoidal and U has a right inverse W, then UF has an augmentation $e\colon UF \to \mathrm{id}$ and $UF \cong (UF(\mathbb{1}) \otimes -)$ as Hopf monads.



For the adjunctions

$$\Delta_!\colon \mathrm{Rep}(G) \leftrightarrows \mathrm{Rep}(G \times G)\colon \Delta^* \quad \text{and} \quad \Delta_!\colon D(X) \leftrightarrows D(X \times X)\colon \Delta^*$$

we can take W to be either π_1^* or π_2^*: we have two augmentations!

The payoff

We thus find that both

$$\Delta^*\Delta_!(\mathbb{C}) \cong \mathbb{C}G^{\mathrm{ad}} \in \mathsf{Rep}(G)$$

and

$$\Delta^*\Delta_!(\mathcal{O}_X) \cong \left(\Delta^*\Delta_*(\mathcal{O}_X)\right)^\vee \cong \Delta^!\Delta_*(\mathcal{O}_X) = \mathcal{U} \in D(X)$$

are Hopf algebras which act on the objects in their respective categories.

We can be write down the structure explicitly.