Traces in low-dimensional algebra

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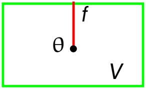
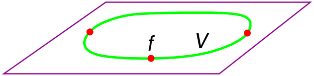
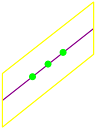
University of Sheffield

$\mathrm{Tr}^{\searrow}(f) := \left\{\right.$  $\left.\right\}$

$\mathrm{Tr}^{\circlearrowleft}(f) :=$ 

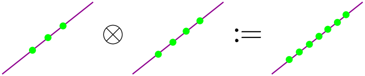
2-tangles: objects

Objects (0-cells)

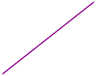


n points on a fixed line in $\mathbb{R}^2$

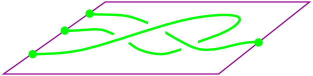
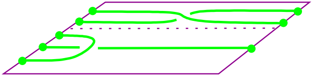
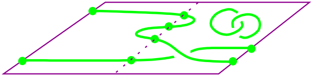
"$\otimes$"-product



unit

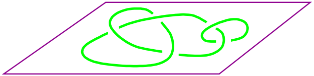


2-tangles: morphisms

Morphisms (1-cells)		tangles in $\mathbb{R}^3$ from m points to n points
Two "products"		$\otimes$ inherited from 0-cells
		$\circ$ composition

Note:

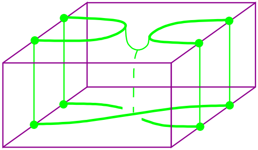
$$\text{1-Hom}(0,0) = \{\text{links}\} = \left\{ \quad \right\}$$



Here both $\otimes$ and $\circ$ are the same.

2-tangles: 2-morphisms

2-morphisms (2-cells)



cobordisms in $\mathbb{R}^4$ from tangle T_1 to tangle T_2

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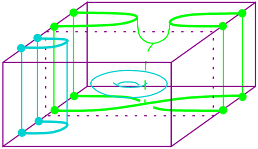
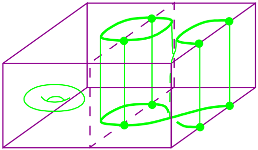
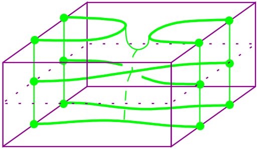
three “products”



$\otimes$ inherited from 0-cells

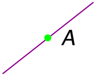
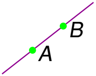
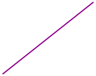


$\circ$ composition from 1-cells



$\circ_2$ composition

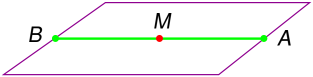
Bimodules: objects

objects (0-cells)	unital associative algebras $/\mathbb{C}$	
"$\otimes$" product	$A \otimes_{\mathbb{C}} B$	
unit	$\mathbb{C}$	

Bimodules: morphisms

morphisms (1-cells)

$1\text{-Hom}_{\mathrm{Bim}}(A,B) :=$ B-A-bimodules ${}_B M_A$



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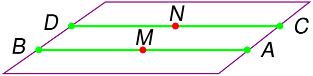
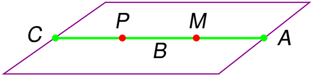
two 'products'

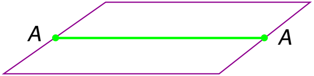
$\otimes$: ${}_B M_A \otimes_{\mathbb{C}} {}_D N_C$
$= {}_{B\otimes D}(M \otimes_{\mathbb{C}} N)_{A\otimes C}$

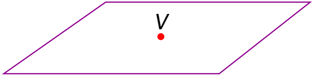


$\circ$: ${}_C P_B \otimes_B {}_B M_A$



Note:

- $\circ$-unit: ${}_A A_A$ 

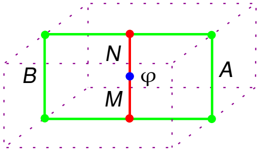
$[{}_M B_A \otimes_A {}_A A_A \cong {}_M B_A]$
- $1\text{-Hom}(\mathbb{C},\mathbb{C}) = \{\text{vector spaces}\} = \{$ 

$\}$

Bimodules: 2-morphisms

2-morphisms (2-cells)

2-$\mathrm{Hom}_{\mathrm{Bim}}(A, B) :=$ bimod maps ${}_B M_A \xrightarrow{\varphi} {}_B N_A$



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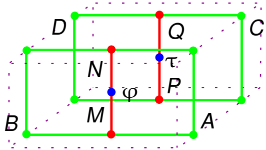
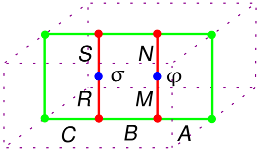
three 'products'

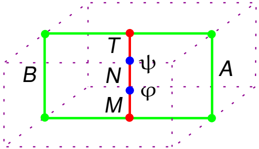
$\otimes$



$\circ$



$\circ_2$



Remarks

- This gives an enlargement of the usual category of algebras and algebra morphisms: $(f\colon A \to B) \mapsto {}_B B_{fA}$
- We have monoidal 2-categories

  $$\text{Bim}{:=}\{\text{algebras, bimodules, bimodule maps}\}$$
  $$\text{Cat}{:=}\{\text{categories, functors, natural transformations}\}$$

  There is a 2-functor Bim $\to$ Cat

  $$\begin{aligned} A &\mapsto \text{Rep}(A) \\ {}_B M_A &\mapsto \left({}_B M_A \otimes - \colon \text{Rep}(A) \to \text{Rep}(B)\right) \\ \varphi &\mapsto \text{`obvious' natural transformation} \end{aligned}$$

  Bim is the sensible place in which to do Morita theory.
- Extended TQFTs are often thought of as 2-functors to Cat but frequently factor through Bim.

A derived version DBim

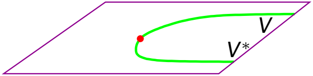
Even more interesting is DBim

objects	(graded) algebras
morphisms	complexes of (graded) bimodules
2-morphisms	morphisms of complexes with quasi-isomorphisms inverted

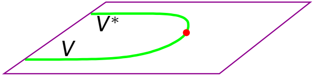
$$\text{2-Hom}_{\mathrm{DBim}}({}_B M_A, {}_B N_A) \cong \mathrm{Ext}^{\bullet}({}_B M_A, {}_B N_A)$$

Duals in a monoidal (2-)category

In $(\mathcal{C}, \otimes, \mathbf{1})$, an object V^* is left-dual to V if there exist morphisms

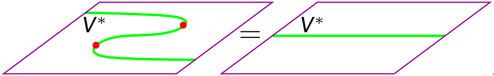


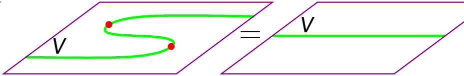
$\mathbf{1} \xleftarrow{\text{ev}} V^* \otimes V$



$V \otimes V^* \xleftarrow{\text{coev}} \mathbf{1}$

such that





If V is also left dual to V^* then V and V^* are bidual.

[$(\text{Vect}, \otimes, \mathbb{C})$ If V is fin dim then $\text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is bidual to V.]

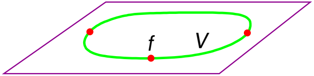
2-Tang Every object is self bidual.

Bim A is bidual to A^{op} via ${}_{\mathbb{C}}A_{A^{\text{op}} \otimes A}$ and ${}_{A \otimes A^{\text{op}}}A_{\mathbb{C}}$

DBim Similar

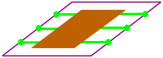
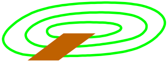
The round trace in a monoidal (2-)category

If V has a bidual and $V \overset{f}{\leftarrow} V$ define the round trace

$\mathrm{Tr}^{\circlearrowright}(f) :=$ 

$\in \mathrm{Hom}(\mathbf{1}, \mathbf{1})$.

[Vect $\mathrm{Tr}^{\circlearrowright}(f) = \big(- \times \mathrm{Tr}(f)\big) \colon \mathbb{C} \to \mathbb{C}$]

2-Tang $\mathrm{Tr}^{\circlearrowright}\Bigg($  $\Bigg) =$ 

Bim $\mathrm{Tr}^{\circlearrowright}\big({}_A M_A\big) = {}_A A_A \otimes_{A \otimes A^{\mathrm{op}}} {}_A M_A \cong M/\{am = ma\}$

DBim $\mathrm{Tr}^{\circlearrowright}\left({}_A M_A\right) = {}_A A_A \otimes^{\mathsf{L}}_{A \otimes A^{\mathrm{op}}} {}_A M_A \cong \mathrm{HH}_\bullet(A, {}_A M_A)$

Motivation/Application: Khovanov homology

Jones:

$$\{\text{links}\} \xrightarrow{\text{Jones}} \mathbb{Z}[q^{\pm 1}]$$

$$\longmapsto -q^{-9} + q^{-5} + q^{-2} + 1$$

Khovanov:

$$\{\text{links}\} \xrightarrow{\text{KH}} \{\text{bigraded vector sps}\} \xrightarrow{\chi} \mathbb{Z}[q^{\pm 1}]$$

$$\bigoplus_{i,j} V^{i,j} \mapsto \sum_{i,j} (-1)^i q^j \dim V^{i,j}$$

Functoriality of Khovanov homology:

$$\big(\text{cobordism } C\colon K \rightsquigarrow K'\big) \mapsto \big(\text{linear map } \text{KH}(C)\colon \text{KH}(K) \to \text{KH}(K')\big)$$

Motivation/Application: Khovanov homology (ctd)

Rouquier:

$$\{A_n\}_{n\in\mathbb{N}} \qquad \text{algebras } A_0 = \mathbb{C}$$

$$n,n\text{-braid} \quad \beta \mapsto {}_{A_n}M(\beta)_{A_n} \quad \text{complex of bimods}$$

$$M(\beta_2 \circ \beta_1) \cong M(\beta_2) \otimes_{A_n} M(\beta_1)$$

$\widetilde{\mathrm{KH}}\Big($  $\Big) \cong \mathrm{HH}_\bullet\Big(A_n, M\Big($  $\Big)\Big)$

This ‘suggests’ that Khovanov homology extends to a monoidal 2-functor

2-Tang $\longrightarrow$ DBim

$n \longmapsto A_n$

m, n-tangle $\longmapsto$ cplx of graded A_n-A_m-bimods

[link $\longmapsto$ cplx of graded vector sps $\mapsto$ KH(link)]

tangle cobordism $\longmapsto$ morphism of complexes

2-characters of finite groups

Finite group G acting on V a vector space: $V \xrightarrow{\rho(g)} V$

$$\mathrm{ch}_\rho(g) := \mathrm{Tr}\big(\rho(g)\big) \in \mathbb{C}; \qquad \mathrm{ch}_\rho(g) = \mathrm{ch}_\rho(hgh^{-1})$$

Finite group G acting on $\mathcal{V}$ a semisimple linear category: $\mathcal{V} \xrightarrow{\alpha(g)} \mathcal{V}$

$$\mathrm{Ch}_\alpha(g) := \mathrm{Nat}\big(\mathrm{Id}_\mathcal{V}, \alpha(g)\big) \in \mathrm{Vect}; \qquad \mathrm{Ch}_\alpha(g) \xrightarrow{\simeq} \mathrm{Ch}_\alpha(hgh^{-1})$$

[Gives a representation of the Drinfeld double of G.]

$\alpha(g)$ is a 1-endomorphism in the 2-category

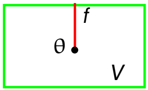
$$\{\text{linear categories, linear functors, natural transformations}\}$$

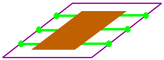
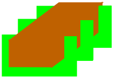
This leads us to another notion of trace…

The diagonal trace in a 2-category

This can be defined in a 2-category without monoidal structure.

If V is an object of a 2-category and $V \xleftarrow{f} V$ define the diagonal trace:

$\mathrm{Tr}^{\searrow}(f) := \text{2-Hom}(\mathrm{Id}_V, f) = \Big\{$  $\Big\}$

2-Tang $\mathrm{Tr}^{\searrow}\Big($  $\Big) = \Big\{$  $\Big\}$

Bim $\mathrm{Tr}^{\searrow}({}_AM_A) = \mathrm{Hom}({}_AA_A, {}_AM_A) \cong \{m \in M \mid am = ma\ \forall a \in A\}$

DBim $\mathrm{Tr}^{\searrow}({}_AM_A) = \mathrm{Ext}^{\bullet}({}_AA_A, {}_AM_A) \cong \mathrm{HH}^{\bullet}(A, {}_AM_A)$