Magnitude and other measures of metric spaces

Simon Willerton
University of Sheffield

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the mathematics of biodiversity
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Overview

category theory
Overview

category theory

Leinster

sets with distance
Overview

category theory

sets with distance

Leinster

geometry

Leinster

Willerton Meckes
Overview

category theory

sets with distance

diversity measures

Leinster

geography

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Willerton

Meckes

Leinster

Cobbold
Overview

- category theory
- sets with distance
- diversity measures
- geometry

Authors: Leinster, Cobbold, Willerton, Meckes
“set with distances” = “metric space”

We have

- a set of ‘points’
- some notion of distance $0 \leq d_{ij} \leq \infty$ between the $i$th and $j$th points.

Note: not every metric space can be thought of as points in Euclidean space.
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For example:

![Diagram showing distances between points Q, SP, NP, and S. The distances are labeled as follows: Q to NP 6, Q to SP 6, Q to S 12, NP to SP 6, NP to S 6, SP to S 12.]
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Magnitude [Leinster]

Metric space $X$ with similarity matrix $Z_{ij} := e^{-d_{ij}}$. 

Define 'weight' (if possible) $-\infty < w_i < \infty$ at each point $i$ so that 

$$
\sum_{j} Z_{ij} w_j = 1 \text{ for every } j
$$

Define the magnitude by 

$$
|X| = \sum_{i} w_i
$$

If $Z_{ij}$ is invertible then 

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|X| = \sum_{ij} (Z_{ij} - 1)_{ij}
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$|X| \sim 1.47$
Example of scaling

As any space $X$ is scaled bigger and bigger, $X \to \mathbb{N}$. 
Example of scaling

As any space $X$ is scaled bigger and bigger, $X \rightarrow N$. 

\[
\begin{array}{c}
|X| \\
0 & 1 & 2 & 3
\end{array}
\]

\[
\begin{array}{c}
t \\
0.0001 & 0.001 & 0.01 & 0.1 & 1 & 10 & 100
\end{array}
\]
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Example of scaling

As any space $X$ is scaled bigger and bigger $|X| \to N$. 
Example of bad metric space

Many metric spaces are better behaved than this. If $Z$ is positive definite then $|X|$ is defined. For example, if $X$ is a subset of Euclidean space then $|X|$ is defined.
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Diversity measures [Leinster, Cobbold]

Model our community using

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- a probability (or relative abundance) $p_i$ at the $i$th point.
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Effective number of species:

$$q D^Z(p) := \begin{cases} 
\left( \sum_{i: p_i > 0} p_i (Zp)_i^{q-1} \right)^{\frac{1}{1-q}} & q \neq 1, \\
\prod_{i: p_i > 0} (Zp)_i^{-p_i} & q = 1, \\
\min_{i: p_i > 0} \frac{1}{(Zp)_i} & q = \infty.
\end{cases}$$
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Recover various other measures of diversity using this. For example, obtain Hill numbers when $d_{ij} = \infty$ (i.e. $Z_{ij} = 0$) for $i \neq j$. 

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Theorem

Let $X$ be a symmetric metric space. So $Z$ is symmetric.
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- If $Z$ is positive definite and there is a weighting with non-negative weights ($w_i \geq 0$), then

$$D_{\text{max}}(Z) = |X|$$

i.e., the magnitude is the maximum diversity for all $q$, and normalizing the weights gives the maximizing probability distribution

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- **Otherwise**

  $$D_{\text{max}}(Z) = \max_{Y \subset X \& w_i > 0} |Y|.$$
Summary of magnitude $|X|$

- Mathematically natural (if mysterious), c.f. category theory.
- Related to biodiversity.
- Seemingly related to geometry in Euclidean space.
- Can behave rather weirdly at times.
Other size measures of metric spaces

- Get Hill Numbers by giving a probability space a dull metric.
- Get numbers for a metric space by giving a dull probability distribution.

\[ E(X) := D_Z((1, \ldots, 1)) \]

For example, analogue of species richness:

\[ E(X) := N \sum_{i=1}^{N} \left( N \sum_{j=1}^{N} Z_{ij} \right) - 1 \]

Note: this is not the same as \[ |X| = N \sum_{i=1}^{N} \sum_{j=1}^{N} (Z_{ij} - 1) \]
Other size measures of metric spaces

- Get Hill Numbers by giving a probability space a dull metric.
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\[ qE(X) := qD^Z \left( \left( \frac{1}{N}, \ldots, \frac{1}{N} \right) \right) \]
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\[ 1000t \]

\[ t \]

\[ 1000t \]

\[ E_0(X) \]

\[ |X| \]
Example of bad metric space II

The size $\mathcal{E}(X)$ is defined for all metric spaces. As $X$ is scaled up, $\mathcal{E}(X)$ increases from 1 to $N$. It is much easier to calculate $\mathcal{E}(X)$ than $|X|$. 
Example of bad metric space II

The size $|E(X)|$ is defined for all metric spaces. As $X$ is scaled up, $|E(X)|$ increases from 1 to $N$. It is much easier to calculate $|E(X)|$ than $|X|$. 
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The size $0^n(E(X))$ is defined for all metric spaces. As $X$ is scaled up, $0^n(E(X))$ increases from 1 to $N$. It is much easier to calculate $0^n(E(X))$ than $|X|$. 
Example of bad metric space II

▶ The size $^0E(X)$ is defined for all metric spaces.
▶ As $X$ is scaled up $^0E(X)$ increases from 1 to $N$.
▶ It is much easier to calculate $^0E(X)$ than $|X|$.
Zooming in on a space with 6400 points
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In a metric space we can scale all the distances. What should happen to the size?
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For example, double the distances:
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1 2
2 4

Think of dimension as how the size changes when the distances are changed. Given 'size' can see if it gives a good idea of dimension.
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\[ \text{2 times as big} \]
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\[ 2^1 = 2 \text{ times as big} \]

\[ 2^2 = 4 \text{ times as big} \]
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Size of rectangles with 6400 points

![Graph showing the size of rectangles with 6400 points as a function of interpoint distance. The x-axis represents the interpoint distance, ranging from 0.00001 to 10, and the y-axis represents the size, ranging from 0 to 6400. A curve illustrates the relationship, indicating that the size increases significantly as the interpoint distance increases.]
Size of rectangles with 6400 points

interpoint distance

- 10 × 640
- 80 × 80
- 1 × 6400
Rectangles with 6400 points and ‘dimension’

There is geometric information is \( 0 \leq E \leq 10 \times 640 \).
Rectangles with 6400 points and ‘dimension’

There is geometric information in $E(X)$. 

![Graph showing growth rate of $E$ vs. interpoint distance for different rectangles.](image)
Rectangles with 6400 points and ‘dimension’

There is geometric information is $^0E(X)$. 