Dualities, enriched categories and metric spaces

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Dualities and relations

Consider the following classical dualities.

- $\{\text{algebraic sets in } \mathbb{C}^n\} \cong \{\text{radical ideals in } \mathbb{C}[x_1, \ldots, x_n]\}^{\mathrm{op}}$
- $\{\text{intermediate extensions } K \subset J \subset L\} \cong \{\text{subgroups of } \mathrm{Gal}(L, K)\}^{\mathrm{op}}$
- $\{\text{closed convex sets in } \mathbb{R}^n\} \cong \{\text{`closed' sets of half spaces in } \mathbb{R}^n\}^{\mathrm{op}}$
- $\{\text{upper closed subsets of } \mathbb{Q}\} \cong \{\text{lower closed subsets of } \mathbb{Q}\}^{\mathrm{op}} \quad [\cong \mathbb{R}]$

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- $\{\text{upper closed subsets of } \mathbb{Q}\} \cong \{\text{lower closed subsets of } \mathbb{Q}\}^{\mathrm{op}} \quad [\cong \mathbb{R}]$

These all arise from a specified relation $I \subset G \times M$ between sets G and M.

We get maps between the ordered sets of subsets

$$\mathcal{P}(G) \leftrightarrows \mathcal{P}(M)^{\mathrm{op}}$$

Restricts to an ordered isomorphism on the 'closed' subsets.

$$\mathcal{P}_{\mathrm{cl}}(G) \cong \mathcal{P}_{\mathrm{cl}}(M)^{\mathrm{op}}$$

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$$G = \mathbb{C}^n,\ M = \mathbb{C}[x_1, \ldots, x_n], \quad xIp \text{ iff } p(x) = 0.$$

- $\{\text{intermediate extensions } K \subset J \subset L\} \cong \{\text{subgroups of } \mathrm{Gal}(L, K)\}^{\mathrm{op}}$

$$G = L,\ M = \mathrm{Aut}(L, K), \quad \ell I \varphi \text{ iff } \varphi(\ell) = \ell.$$

- $\{\text{closed convex sets in } \mathbb{R}^n\} \cong \{\text{`closed' sets of half spaces in } \mathbb{R}^n\}^{\mathrm{op}}$

$$G = \mathbb{R}^n,\ M = \{\text{half spaces in } \mathbb{R}^n\}, \quad xIH \text{ iff } x \in H.$$

- $\{\text{upper closed subsets of } \mathbb{Q}\} \cong \{\text{lower closed subsets of } \mathbb{Q}\}^{\mathrm{op}} \quad [\cong \mathbb{R}]$

$$G = \mathbb{Q},\ M = \mathbb{Q}, \quad qIp \text{ iff } q \leq p.$$

These all arise from a specified relation $I \subset G \times M$ between sets G and M.

$$\mathcal{P}_{\mathrm{cl}}(G) \cong \mathcal{P}_{\mathrm{cl}}(M)^{\mathrm{op}}$$

Formal concept analysis

Here

$$G = \text{some set of objects}, \quad M = \text{some set of attributes}$$

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	needs water to live	lives in water	lives on land	needs chlorophyll	dicotyledon	monocotyledon	can move	has limbs	breast feeds
fish leech	×	×					×		
bream	×	×					×	×	
frog	×	×	×				×	×	
dog	×		×				×	×	×
water weeds	×	×		×		×			
reed	×	×	×	×		×			
bean	×		×	×	×				
corn	×		×	×		×			

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The elements of 'this' poset are called formal concepts.

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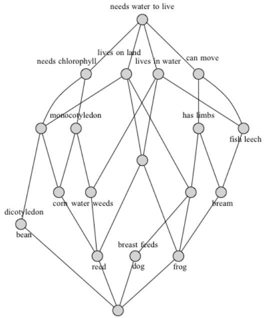

Monoidal categories

A monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$ consists of a category $\mathcal{V}$ with a monoidal product $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and unit $\mathbb{1} \in \mathrm{Ob}(\mathcal{V})$, together with appropriate associativity and unit constraints.

category	objects	morphisms	$\otimes$	$\mathbb{1}$
Set	sets	functions	$\times$	$\{*\}$
Top	topological spaces	continuous maps	$\times$	$\{*\}$
Vect	vector spaces	linear maps	$\otimes$	$\mathbb{C}$
$\overline{\mathbb{R}_+}$	$[0, \infty]$	$a \to b$ iff $a \geq b$	$+$	0
Truth	$\{\mathrm{T}, \mathrm{F}\}$	$a \to b$ iff $a \Rightarrow b$	&	T

Enriched category

A category $\mathcal{C}$ consists of a set $\mathrm{Ob}(\mathcal{C})$ together with

- for each $a, b \in \mathrm{Ob}(\mathcal{C})$ a specified set
$$\mathcal{C}(a, b)$$
- for each $a, b, c \in \mathrm{Ob}(\mathcal{C})$ a function
$$\circ_{a,b,c} \colon \mathcal{C}(a, b) \times \mathcal{C}(b, c) \to \mathcal{C}(a, c)$$
- for each $a \in \mathrm{Ob}(\mathcal{C})$ an element
$$\mathrm{id}_a \in \mathcal{C}(a, a)$$

satisfying appropriate associativity and identity constraints.

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Enriched category

A $\mathcal{V}$-category $\mathcal{C}$ consists of a set $\mathrm{Ob}(\mathcal{C})$ together with

- for each $a, b \in \mathrm{Ob}(\mathcal{C})$ a specified object

$$\mathcal{C}(a, b) \in \mathrm{Ob}(\mathcal{V})$$

- for each $a, b, c \in \mathrm{Ob}(\mathcal{C})$ a morphism in $\mathcal{V}$

$$\circ_{a,b,c} \colon \mathcal{C}(a, b) \otimes \mathcal{C}(b, c) \to \mathcal{C}(a, c)$$

- for each $a \in \mathrm{Ob}(\mathcal{C})$ a morphism in $\mathcal{V}$

$$\mathrm{id}_a \colon \mathbb{1} \to \mathcal{C}(a, a)$$

satisfying appropriate associativity and identity constraints.

Examples of types of enriched categories

$\mathcal{V}$	$\mathcal{C}(a, b)$	composition	identity
Set	set	$\mathcal{C}(a,b) \times \mathcal{C}(b,c) \to \mathcal{C}(a,c)$	$\{*\} \to \mathcal{C}(a,a)$
Top	space	$\mathcal{C}(a,b) \times \mathcal{C}(b,c) \to \mathcal{C}(a,c)$	$\{*\} \to \mathcal{C}(a,a)$
$\overline{\mathbb{R}_+}$	$[0, \infty]$	$\mathcal{C}(a,b) + \mathcal{C}(b,c) \geq \mathcal{C}(a,c)$	$0 \geq \mathcal{C}(a,a)$
Truth	$\{\mathrm{T}, \mathrm{F}\}$	$\mathcal{C}(a,b) \mathbin{\&} \mathcal{C}(b,c) \Rightarrow \mathcal{C}(a,c)$	$\mathrm{T} \Rightarrow \mathcal{C}(a,a)$

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An $\overline{\mathbb{R}_+}$-category is a generalised metric space: write $d(a, b) := \mathcal{C}(a, b)$.
[Fails to be a metric space as $d(a, b) \neq d(b, a)$.]

A Truth-category is a preorder: write $a \leq b$ iff $\mathcal{C}(a, b) = \mathrm{T}$.
[Fails to be a poset as $(a \leq b) \,\&\, (b \leq a) \not\Rightarrow a = b$.]

More structure

$\mathcal{V}$	$\mathcal{V}$-functor	$\mathcal{C} \to \mathcal{V}$	$\mathcal{C} \otimes \mathcal{D}^{\mathrm{op}} \to \mathcal{V}$
Set	functor	copresheaf	profunctor
$\overline{\mathbb{R}_+}$	distance non-increasing map	$X \to [0, \infty]$	cost function
Truth	order-preserving function	lower closed subset	relation

Even more structure

When $\mathcal{V}$ is particularly nice we can define $[\mathcal{C}, \mathcal{V}]$ a $\mathcal{V}$-category structure on the collection of $\mathcal{V}$-functors $\mathcal{C} \to \mathcal{V}$.

- $\mathcal{V} = \mathrm{Set}$
  objects are functors $\mathcal{C} \to \mathrm{Set}$.
  $[\mathcal{C}, \mathrm{Set}](F, G) :=$ natural transformations F to G

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- $\mathcal{V} = \overline{\mathbb{R}_+}$
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  $d(F, G) := \sup_c(G(c) - F(c))$

- $\mathcal{V} = \mathrm{Truth}$
  objects are upward closed subsets
  $P \leq Q$ iff $P \subseteq Q$

Generalizing the relation-to-duality idea

- $\mathcal{V}$, suitable category to enrich over,
- $\mathcal{C}$, a $\mathcal{V}$-category,
- $\mathcal{D}$, a $\mathcal{V}$-category,
- $I \colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathcal{V}$ a profunctor from $\mathcal{C}$ to $\mathcal{D}$.

Get an adjunction of $\mathcal{V}$-categories

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \leftrightarrows [\mathcal{D}, \mathcal{V}]^{\mathrm{op}}$$

which restricts to an equivalence of $\mathcal{V}$-categories

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]_{\mathrm{cl}} \cong [\mathcal{D}, \mathcal{V}]^{\mathrm{op}}_{\mathrm{cl}}.$$

We can think of this as a single $\mathcal{V}$-category $\mathcal{B}(\mathcal{C}, \mathcal{D}, I)$.
This is called the profunctor nucleus [Pavlovic].

Example 0: Classical Galois connections

- $\mathcal{V} = \mathrm{Truth}$,
- $\mathcal{C} = G$, a set
- $\mathcal{D} = M$, a set
- I a relation between G and M.

Get the construction of an isomorphism of posets from a relation

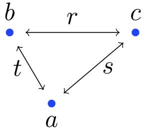
$$\mathcal{P}_{\mathrm{cl}}(G) \cong \mathcal{P}_{\mathrm{cl}}(M)^{\mathrm{op}}$$

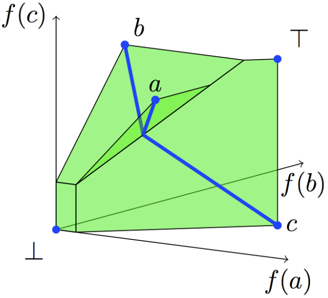
We can think of this as a single poset $\mathcal{B}(G, M, I)$.
This gives all of the classical examples from the beginning.

Example 1: Directed tight span

- $\mathcal{V} = \overline{\mathbb{R}_+}$,
- $\mathcal{C}$ = a metric space,
- $\mathcal{D} = \mathcal{C}$,
- $I(c, c') := d(c, c')$.

The generalized metric space $\mathcal{B}(\mathcal{C}, \mathcal{C}, d)$ is the directed tight span of $\mathcal{C}$.

$\mathcal{C} =$ 

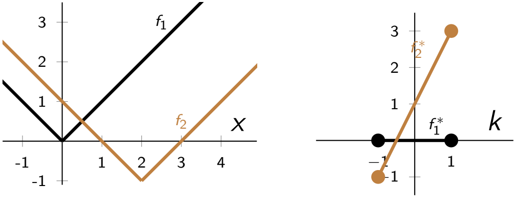
$\mathcal{B}(\mathcal{C}, \mathcal{C}, d) =$ 

Example 2: Legendre-Fenchel transform

- $\mathcal{V} = \overline{\mathbb{R}}$,
- $\mathcal{C} = \mathbb{R}^n$,
- $\mathcal{D} = (\mathbb{R}^n)^\vee$ the dual space,
- $I(x, k) := k(x)$.

Maps of generalized metric spaces: Legendre-Fenchel transform

$$\{\text{functions on } \mathbb{R}^n\} \leftrightarrows \{\text{functions on } (\mathbb{R}^n)^\vee\}^{\text{op}}$$



$$\{\text{convex functions on } \mathbb{R}^n\} \cong \{\text{convex functions on } (\mathbb{R}^n)^\vee\}^{\text{op}}$$

Example 3: Fuzzy concept analysis

- $\mathcal{V} = ([0,1], \cdot, 1)$, thought of as fuzzy truth values,
- $\mathcal{C} = \{\text{objects}\}$,
- $\mathcal{D} = \{\text{attributes}\}$,
- $I(g, m) \in [0,1]$, degree to which object g has an attribute m.

The resulting fuzzy poset is the fuzzy concept lattice.

Example 4: Reflexive modules

- $\mathcal{V} = \mathrm{Ab}$, the category of Abelian groups,
- $\mathcal{C}$, a one object Ab-category,
- $\mathcal{D} = \mathcal{C}$,
- $I \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Ab}$ is the corresponding ring R.

The adjunction is formed from the duality map $\mathrm{Hom}(-, R)$:

$$\{\text{left } R\text{-modules}\} \leftrightarrows \{\text{right } R\text{-modules}\}^{\mathrm{op}}.$$

The nucleus is

$$\{\text{reflexive left } R\text{-modules}\} \cong \{\text{reflexive right } R\text{-modules}\}^{\mathrm{op}}.$$